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ON COMMUTING LINEAR PARTIAL DIFFERENTIAL OPERATORS

By

J. HAINZL (Kassel)

Introduction and notations. Let Ω be an open connected subset of the real n-dimensional space $\mathbf{R}^n$, $x=(x_1, \dots, x_n)\in\Omega$, $D_i=\dfrac{\partial}{\partial x_i}$, and for any n-tuple $\alpha=(\alpha_1, \dots, \alpha_n)$ of nonnegative integers α_i (for short: $\alpha\in\mathrm{N}_0^n$) let D^α denote the operator $D_1^{\alpha_1} \dots D_n^{\alpha_n}$, and put $|\alpha|:=\alpha_1+\dots+\alpha_n$. We consider linear homogeneous partial differential operators L of any order $m\in\mathrm{N}_0$:

$$L:=\sum_{|\alpha|\leqq m} A_\alpha(x)\,D^\alpha \tag{1}$$

where A_α are assumed to be complex-valued C^∞-functions defined on Ω. If P is another such operator, we say that P commutes with L if $PL-LP=0$. We shall prove results on commutability of L with more special operators P, such as powers of operators having constant coefficients or monomial operators $P:=D_1^{p_1}\dots D_n^{p_n}$. The methods of proof are elementary; nevertheless, the proof of Theorem 2 is rather lengthy. Some of the results generalize theorems proved in [1], and Theorem 1 is stated in [1] without proof. Further references concerning the topic of this note are not known to the author.

THEOREM 1. *Assume L to commute with $M:=D_1^{p_1}\dots D_n^{p_n}$, where p_k are fixed nonnegative integers. Then, for any k with $p_k>0$ and any $\alpha=(\alpha_1, \dots, \alpha_n)$, the coefficient $A_\alpha(x)$, regarded as a function of x_k only, is a polynomial of degree not exceeding α_k. In particular, if $p_k>0$ for each k, the coefficients of L are polynomials.*

PROOF. A straightforward calculation yields

$$0=ML-LM=\sum_{|\alpha|\leqq m}\sum_{\delta} c(\delta)\cdot(D^\delta A_\alpha)(x)\cdot D^{\alpha+\bar\delta}. \tag{2}$$

Here, the inner sum has to be taken over all $\delta=(\delta_1, \dots, \delta_n)\in\mathrm{N}_0^n$ such that $|\delta|>0$ and $0\leqq\delta_i\leqq p_i$. Moreover, $\bar\delta:=(p_1-\delta_1, \dots, p_n-\delta_n)$ and $c(\delta)$ is the product $\binom{p_1}{\delta_1}\cdot\ldots\cdot\binom{p_n}{\delta_n}$ of binomial coefficients. Now, consider the coefficient $C_\gamma(x)$ of D^γ in (2) for any $\gamma=(\gamma_1, \dots, \gamma_n)\in\mathrm{N}_0^n$ satisfying $|\gamma|\leqq m-1+p_1+\dots+p_n$, $0\leqq\gamma_i\leqq m+p_i$. C_γ must vanish identically and is given by

$$0=C_\gamma(x)=\sum_{\delta} c(\delta)\,D^\delta A_{\gamma+\delta-p}(x), \quad \text{where} \quad p:=(p_1, \dots, p_n). \tag{3}$$

(Here, A_α is understood to be zero if $|\alpha|>m$ or $\alpha_i<0$ for some index i.) The assertion concerning A_α will be proved by induction on $|\alpha|=m, m-1, \dots, 0$. Without loss of generality, we may assume $p_i>0$ for $i=1, \dots, r$ and $p_i=0$ for $i>r$ ($r\geqq 1$). In what follows, we say that property $P(\alpha, k)$ holds iff A_α, regarded as a function of x_k only, is a polynomial of degree not exceeding α_k. First we show:

(i) For any α, k with $|\alpha|=m$, $1\leqq k\leqq r$, property $P(\alpha, k)$ holds. This is proved by induction on $\alpha_k=0, 1, \dots, m$. Denoting by ε_i the n-tuple $(0, \dots, 0, 1, 0, \dots, 0)$ with 1 in position i, we choose $\gamma=\alpha+p-\varepsilon_k$ in (3), where α and k are fixed, $|\alpha|=m$. Then δ is restricted to the values $\delta=\varepsilon_i$, $1\leqq i\leqq r$; so we get

$$p_k D_k A_\alpha = -\sum_{\substack{i=1\\ i\neq k}}^{r} p_i D_i A_{\alpha+\varepsilon_i-\varepsilon_k}. \tag{4}$$

If $\alpha_k=0$, we have $A_{\alpha+\varepsilon_i-\varepsilon_k}=0$ for $i\neq k$, hence $p_k D_k A_\alpha=0$; that is, A_k does not depend on x_k and, thus, assertion (i) is true for $\alpha_k=0$, $1\leqq k\leqq r$.

Now assume that, with some t, $0\leqq t\leqq m-1$, property $P(\alpha, k)$ is already proved for $k=1, \dots, r$ and any α satisfying $|\alpha|=m$, $\alpha_k=t$. Then we have to prove $P(\alpha, k)$ for any k and any α such that $|\alpha|=m, \alpha_k=t+1$. Look again at equation (4). Putting $\beta:=\alpha+\varepsilon_i-\varepsilon_k$ for any $i\neq k$, we have $|\beta|=m$ and $\beta_k=\alpha_k-1=t$. Thus, by induction hypothesis, A_β is a polynomial with respect to x_k whose degree does not exceed t. Then, the same assertion holds for $D_i A_\beta$, and, consequently, for the right-hand side of (4), too. Hence, A_α, as a function of x_k, is a polynomial of degree $\leqq t+1=\alpha_k$. This concludes the proof of (i).

(ii) Induction hypothesis: For some s, $1\leqq s\leqq m$, assume that property $P(\alpha, k)$ holds for any $k=1, \dots, r$ and any α with $|\alpha|\geqq s$. It remains to show that property $P(\alpha, k)$ also holds for any $k=1, \dots, r$ and any α satisfying $|\alpha|=s-1$. This will again be proved by induction on $\alpha_k=0,1, \dots, m$. For any fixed α with $|\alpha|=s-1$ and any fixed k we choose $\gamma:=\alpha+p-\varepsilon_k$ in (3). This leads to

$$p_k D_k A_\alpha = -\sum_{\substack{i=1\\ i\neq k}}^{r} p_i D_i A_{\alpha+\varepsilon_i-\varepsilon_k} - \sum_{|\delta|\geqq 2} c(\delta)\, D^\delta A_{\alpha+\delta-\varepsilon_k}. \tag{5}$$

Consider the terms in the second sum on the right-hand side of (5). If we put $\beta:=\alpha+\delta-\varepsilon_k$, from $|\alpha|=s-1$ we get $|\beta|\geqq s$. So, by induction hypothesis (ii), property $P(\beta, i)$ holds for $i=1, \dots, r$. Hence, we conclude that $D^\delta A_\beta$, as a function of x_i only, is a polynomial of degree $\leqq \nu_i$, where $\nu:=\alpha-\varepsilon_k$. (If $\nu_i<0$ for some i, this means $D^\delta A_\beta=0$.) This implies that, if we put $\alpha_k=0$ in (5), the right-hand side must vanish. Thus, $D_k A_\alpha=0$, whence property $P(\alpha, k)$ is valid for every $k=1, \dots, r$ and any α with $|\alpha|=s-1$, $\alpha_k=0$.

Now assume for some t, $0\leqq t\leqq s-2$, that property $P(\alpha, k)$ holds for any $k=1, \dots, r$ and any α satisfying $|\alpha|=s-1$, $\alpha_k=t$.

For any fixed k and any fixed α with $|\alpha|=s-1$, $\alpha_k=t+1$, we again consider equation (5). In the first sum on the right-hand side, put $\mu:=\alpha+\varepsilon_i-\varepsilon_k$. Then $|\mu|=s-1$, $\mu_k=\alpha_k-1=t$. Hence, by assumption, $P(\mu, k)$ holds. This implies that, with respect to x_k, $D_i A_\mu$ is a polynomial of degree $\leqq t$. Moreover, by the reasoning above, the second sum in (5), regarded as a function of x_k, is a polynomial of degree $\leqq \nu_k=\alpha_k-1=t$. Consequently, this assertion is also true for $D_k A_\alpha$, whence we conclude $P(\alpha, k)$ to hold for any $k=1, \dots, r$ and any α satisfying $|\alpha|=s-1$, $\alpha_k=t+1$. Now, both inductions are completed and thus, Theorem 1 is proved.

REMARKS. 1. The converse of Theorem 1 does not hold. For example, by Theorem 3c) below, a necessary condition for $D_1^{p_1}$ ($p_1>0$) to commute with L is that any A_α, as a function of x_1 only, be a polynomial of degree $\leqq 0$, not merely of degree $\leqq \alpha_1$.

2. If $ML=LM$, M as in Theorem 1 with $p_1>0$, $p_2>0$, then the degrees d_1, d_2 of A_α belonging to the variables x_1, x_2 respectively, are not independent of each other. For instance, if $M=D_1^{p_1}D_2^{p_2}$ and $d_1=0$ for each α, then also $d_2=0$ for each α. This is a simple consequence of Theorem 3c) below.

3. Theorem 1 need not remain true if M is replaced by some more general differential operator P with constant coefficients. In case $n=2$ for example, $P:=D_1+D_2$ commutes with any L whose coefficients are arbitrary C^1-functions of x_1-x_2. In the case of ordinary differential operators, however, the following result can be proved (by induction, beginning with the highest coefficient of L): Let P be a linear homogeneous differential operator of order at least 1 with constant coefficients. Then, if P commutes with L, L has constant coefficients. Trivially, the converse is also true.

THEOREM 2. *Assume L to have polynomial coefficients and let P be any linear homogeneous partial differential operator in $\mathbf{R}^n$ with constant coefficients. Then L commutes with P iff L commutes with P^k for at least one positive integer k.*

PROOF. Write P in the form

$$P=\sum_{j=1}^{l} c_j M_j \quad\text{with}\quad M_j=D_1^{p_{j1}}\dots D_n^{p_{jn}},\quad M_i\neq M_j \quad\text{for}\quad i\neq j,\ l\in\mathbf{N},\quad\text{and}\quad c_j\in\mathbf{C}\setminus\{0\}.$$

By suitably renumbering the M_j, the following situation can be obtained: There exist natural numbers r_i, $l=r_0\geqq r_1\geqq\dots\geqq r_n\geqq 1$, such that

$$p_{1i}=p_{2i}=\dots=p_{r_i,i}<p_{r_i+1,i}\leqq\dots\leqq p_{r_{i-1},i} \quad\text{for}\quad 1\leqq i\leqq n. \tag{6}$$

This can easily be proved by induction on $i=1, \dots, n$. Moreover, $r_n=1$ must hold, since otherwise we would have $M_1=M_2$. This and property (6) will be needed in a later part of the proof. Without loss of generality, we may assume $c_1=1$.

Now, for any $k\in\mathbf{N}$, a straightforward calculation yields the identity for $x\in\mathbf{R}^n$:

$$P^k L-LP^k=\sum_t \sum_{|\alpha|\leqq m} \sum_\delta a(k, t)\cdot b(\delta, k, t)\cdot (D^\delta A_\alpha)(x)\cdot D^{\alpha+\bar{\delta}} \tag{7}$$

where the following notations and conventions hold: $\sum_t$ means summation over all $t=(t_2, \dots, t_l)\in\mathbf{N}_0^{l-1}$. (But only finitely many terms of the sum are nonzero.)

$$a(k, t):=\frac{k(k-1)\dots(k-t_2-t_3-\dots-t_l+1)}{t_2!\dots t_l!}\cdot c_2^{t_2}\dots c_l^{t_l}.$$

$\sum_\delta$ means summation over all $\delta=(\delta_1, \dots, \delta_n)\in\mathbf{N}_0^n$, $0<|\delta|\leqq d$, where d is an upper bound for the degrees of all the polynomials A_α. $\delta+\bar{\delta}=\beta(k, t)=(\beta_1(k, t), \dots, \beta_n(k, t))$ with

$$\beta_i(k, t):=p_{1i}k+\sum_{j=2}^{l}(p_{ji}-p_{1i})\,t_j,$$

and $b(\delta, k, t)$ denotes the product of binomial coefficients

$$\prod_{i=1}^{n} \binom{\beta_i(k, t)}{\delta_i}.$$

In case $l=1$, the first sum in (7) has to be dropped, and $a(k, t) :=1$.

In (7), we now determine the coefficient B_γ of D^γ for any fixed $\gamma=(\gamma_1, \dots, \gamma_n)$ with $\gamma_i \geqq (k-1)p_{1i}$. We obtain

$$B_\gamma = \sum_{|\alpha|\leqq m} \sum_{\delta} \sum_{t}{}' a(k, t) \cdot b(\delta, k, t) \cdot (D^\delta A_\alpha)(x), \tag{8}$$

where $\sum'$ means summation over all $t \in \mathrm{N}_0^{l-1}$ satisfying the condition $\beta(k, t)=\gamma+\delta-\alpha$

From now on, we fix k and γ. For any $s \in \mathrm{N}$ we then determine, in the operator P^sL-LP^s, the coefficient of $D^{\gamma(s)}$, where $\gamma(s):=\gamma+(s-k)p_1$, $p_1:=(p_{11}, \dots, p_{1n})$. Denoting this coefficient by $R(s, x)$, we obtain from (7) and (8)

$$R(s, x) = \sum_{|\alpha|\leqq m} \sum_{\delta} \sum_{t}{}'' a(s, t) \cdot b(\delta, s, t) \cdot (D^\delta A_\alpha)(x), \tag{9}$$

where now $\sum''$ means summation over all $t \in \mathrm{N}_0^{l-1}$ satisfying the condition

$$\beta(s, t) = \gamma(s)+\delta-\alpha. \tag{10}$$

It is easily seen that, for any fixed α and δ, these t are just the same as those admitted in (8); that is, $\sum''=\sum'$. Let $T(\alpha, \delta)$ denote this set of $(l-1)$-tuples t. Then $T(\alpha, \delta)$ does not depend on s. In the last part of the proof (see (11) below), we shall show that $T(\alpha, \delta)$ is a finite set. Therefore, if we write

$$Q(s, \alpha, \delta, x) := \sum_{t \in T(\alpha, \delta)} a(s, t) \cdot b(\delta, s, t) \cdot (D^\delta A_\alpha)(x),$$

we conclude from the definitions of $a(s, t)$ and $b(\delta, s, t)$ that $Q(z, \alpha, \delta, x)$ is a polynomial with respect to the variable z. Since

$$R(s, x) = \sum_{|\alpha|\leqq m} \sum_{\delta} Q(s, \alpha, \delta, x)$$

the function $R(z, x)$, too, is a polynomial with respect to z. Now, assume k to have been chosen such that P^k commutes with L. Then $(P^k)^j=P^{kj}$ also commutes with L for any $j \in \mathrm{N}$. This implies $R(s, x)=0$ for $s=kj$, $j \in \mathrm{N}$, and every $x \in \mathrm{R}^n$. Thus, for any fixed x, the polynomial $R(z, x)$ has infinitely many zeros, and, therefore, vanishes identically. Obviously, $R(1, x)=0$ now implies $PL-LP=0$.

Thus, for completing the proof it remains to show:

(11) For any α, δ satisfying $0 \leqq |\alpha| \leqq m$, $0<|\delta| \leqq d$, the set $T(\alpha, \delta)$ is finite.

Let $\bar{t}=(\bar{t}_2, \dots, \bar{t}_l)$ be a fixed and $t=(t_2, \dots, t_l)$ an arbitrary element of $T(\alpha, \delta)$. Using equation (10) and putting $\bar{t}-t=: u=(u_2, \dots, u_l)$, we obtain the following system of linear homogeneous equations

$$\sum_{j=2}^{l} (p_{ji}-p_{1i})u_j = 0, \quad 1 \leqq i \leqq n. \tag{12}$$

First we prove:

$$\begin{cases} \text{Any nontrivial solution } u \in \mathrm{R}^{l-1} \text{ of (12) has at least} \\ \text{one positive and at least one negative component.} \end{cases} \tag{13}$$

Without loss of generality, let u be a nonnegative solution of (12). By relation (6), we then deduce from the first equation of (12): $u_j=0$ for $r_1<j \leqq l$. System (12) now turns into

$$\sum_{j=2}^{r_1} (p_{ji}-p_{1i})\, u_j = 0, \quad 1 \leqq i \leqq n,$$

and choosing $i=2$, we again conclude from (6): $u_j=0$ for $r_2<j \leqq r_1$, etc. Inductively, if i is the smallest index satisfying $r_i=1$, we obtain $u_j=0$ for $j>r_{i-1}$, and

$$\sum_{j=2}^{r_{i-1}} (p_{ji}-p_{1i})\, u_j = 0,$$

whence finally $u_j=0$ for $2=r_i+1 \leqq j \leqq r_{i-1}$. Hence $u=0$, and (13) is proved.

To complete the proof of (11), it is enough to verify the following assertion: For $\bar{t} \in T(\alpha, \delta)$ as above, put

$$K := \max_{2 \leqq j \leqq l} \bar{t}_j$$

and denote by U_K the set of solutions $u=(u_2, \dots, u_l)$ of (12) with u_j integers and $u_j \leqq K$ for $2 \leqq j \leqq l$. Then

$$U_K \text{ is a finite set.} \tag{14}$$

For, assume (14) to be true. Then for any $t \in T(\alpha, \delta)$, we have $t=\bar{t}-u$ for some integer solution u of (12), whose components obviously satisfy $u_j \leqq K$. Hence $u \in U_K$ and, owing to (14), $T(\alpha, \delta)$ must be finite.

To prove (14) we define for any $u \in U_K$, $0 \leqq j \leqq l-1$:

$$V_{u,j} := \{v=(v_2, \dots, v_l) \in U_K \mid u_i \neq v_i \text{ for at most } j \text{ of the indices } i\}.$$

Then $V_{u,l-1}=U_K$ for any u, so it is enough to show for $0 \leqq j \leqq l-1$:

$$V_{u,j} \text{ is a finite set for any } u \in U_K. \tag{15}$$

This is done by induction on $j=0, \dots, l-1$. Since $V_{u,0}=\{u\}$, (15) is true for $j=0$. Now assume (15) to hold for $j=q$, q some fixed integer satisfying $0 \leqq q \leqq l-2$. Then, for any fixed $u \in U_K$, we choose some $v \in V_{u,q+1}$. If $v \neq u$, the solution $u-v$ of (12) has at least $l-q-2$ vanishing components and, by (13), at least one negative component. There are but a finite number of possible positions for these $l-q-1$ components. Hence, we may assume without loss of generality

$$u_i = v_i \quad \text{for} \quad 2 \leqq i \leqq l-q-1, \quad u_{l-q} < v_{l-q}. \tag{16}$$

Since $v_{l-q} \leqq K$, v_{l-q} can adopt only finitely many values. Moreover, by induction hypothesis, there exist only finitely many $w \in U_K$ such that $w_i=v_i$ for $2 \leqq i \leqq l-q$. Consequently, there exist only finitely many $v \in V_{u,q+1}$ satisfying (16). Hence, $V_{u,q+1}$ is a finite set, too. This proves (15), whence the proof of Theorem 2 is complete.

COROLLARY 1. *Let P be a linear homogeneous differential operator with constant coefficients which contains only $D_1, \dots, D_r$ $(r \leqq n)$. Moreover, let the C^∞-coefficients $A_\alpha(x_1, \dots, x_n)$ of L be polynomials with respect to the variables $x_1, \dots, x_r$. Then L commutes with P iff L commutes with P^k for some positive integer k.*

PROOF. Using the notations of the preceding proof, we now have $p_{ji}=0$ for $1 \leqq j \leqq l$ and $r<i \leqq n$, hence $\beta_i(s,t)=0$ for $r<i\leqq n$ and any $s \in \mathbf{N}$. Therefore, summation over δ in (7), (8), (9) may be confined to those $\delta \in \mathbf{N}_0^n$ satisfying $0<|\delta|\leqq d$, $\delta_{r+1}=\dots=\delta_n=0$, where now d is an upper bound for the degrees of all the coefficients A_α, considered as polynomials in $x_1, \dots, x_r$. The rest of the proof coincides with that of Theorem 2.

COROLLARY 2. *Let L possess C^∞-coefficients and let P denote a monomial operator $P:=cD_1^{p_1}\dots D_n^{p_n}$ with $p_i \in \mathbf{N}_0$, $c \in \mathbf{C}$. Then L commutes with P iff L commutes with some positive power P^k.*

PROOF. Immediate from Theorem 1 and Corollary 1. For a direct proof of Corollary 2, see [1], Theorem 6.

REMARK. To indicate limitations for possible generalizations of the preceding results we consider two simple examples. 1) $L:=2x_1D_1-x_2D_2$ commutes with $D_1D_2^2$, but does not commute with D_1D_2. 2) $L:=D_1D_2+x_1D_1-x_2D_2$ commutes with D_1D_2, but does not commute with D_1 or D_2.

The following theorem is proved by essential use of Theorem 2 and its corollaries which themselves turn out to be special cases of this new theorem.

THEOREM 3. a) *Let $P_1, \dots, P_l$ be nonvanishing linear homogeneous differential operators with constant coefficients in which only $D_1, \dots, D_r$ are involved $(r \leqq n)$. Moreover, let all the C^∞-coefficients $A_\alpha(x_1, \dots, x_n)$ of L be polynomials with respect to $x_1, \dots, x_r$. Finally, assume that L commutes with the operators*

$$P_1^{a_{1k}}\dots P_l^{a_{lk}} \quad 1 \leqq k \leqq s, \tag{17}$$

where a_{ik} are fixed nonnegative integers. Then, L also commutes with $P_1^{c_1}\dots P_l^{c_l}$, where $(c_1, \dots, c_l) \in \mathbf{N}_0^l$ is any linear combination (over $\mathbf{R}$) of the s vectors $(a_{1k}, \dots, a_{lk})$. Particularly, if $l \leqq s$ and the matrix (a_{ik}) has rank l, then L commutes with every P_i, $1 \leqq i \leqq l$.

b) *Assume that in* a) *the operators P_j are of the form*

$$P_j := C_jD_1^{p_{1j}}\dots D_n^{p_{nj}}, \; p_{ij} \in \mathbf{N}_0, \; C_j \in \mathbf{C}\setminus\{0\}, \quad 1 \leqq j \leqq l.$$

Let L have arbitrary C^∞-coefficients and, finally, assume L to commute with the operators (17). *Then the assertions from* a) *remain true.*

c) *Assume L to have C^∞-coefficients and to commute with $D_1^{a_{1k}}\dots D_l^{a_{lk}}$, $k=1, \dots, l$, where* $\det(a_{ik}) \neq 0$. *Then L commutes with every D_i, $1 \leqq i \leqq l$. Hence, the coefficients of L do not depend on $x_1, \dots, x_l$.*

PROOF. Let R denote the rank of the matrix $(a_{ik})_{\substack{1\leqq i \leqq l \\ 1 \leqq k \leqq s}}$. Without loss of generality, assume $R>0$ and $\det A \neq 0$ with $A:=(a_{ik})_{\substack{1\leqq i \leqq R \\ 1 \leqq k \leqq R}}$. For any vector $(c_1, \dots, c_l)$

satisfying the assumptions of the theorem, we then get

$$c_i = \sum_{k=1}^{R} t_k a_{ik}, \quad 1 \leqq i \leqq l$$

where $t_k \in \mathbf{R}$ are uniquely determined. More precisely, by Cramer's rule, we have $t_k=u_kd^{-1}$, where $d:=|\det A| \in \mathbf{N}$ and u_k are integers, not all of them negative. Without restriction, we may assume $u_k \geqq 0$ for $1 \leqq k \leqq q$ and $u_k<0$ for $q<k\leqq R$. Denoting by P^{a_k} the operator $P_1^{a_{1k}}\dots P_l^{a_{lk}}$, we find that L commutes with the operators

$$Q_1 := (P^{a_1})^{u_1}\dots(P^{a_q})^{u_q} = P_1^{b_1}\dots P_l^{b_l}$$

and

$$Q_2 := (P^{a_{q+1}})^{|u_{q+1}|}\dots(P^{a_R})^{|u_R|} = P_1^{d_1}\dots P_l^{d_l}$$

where

$$b_i = \sum_{k=1}^{q} a_{ik}u_k, \quad d_i = -\sum_{k=q+1}^{R} a_{ik}u_k, \quad 1 \leqq i \leqq l$$

and

$$b_i-d_i = \sum_{k=1}^{R} a_{ik}u_k = dc_i \geqq 0.$$

Putting

$$Q_3 := P_1^{b_1-d_1}\dots P_l^{b_l-d_l} = (P_1^{c_1}\dots P_l^{c_l})^d,$$

we have $Q_1=Q_2Q_3=Q_3Q_2$. Now, $Q_3Q_2L-LQ_3Q_2=0$, $Q_2L=LQ_2$; hence $Q_3LQ_2-LQ_3Q_2=(Q_3L-LQ_3)Q_2=0$. This yields $L_1:=Q_3L-LQ_3=0$ in the following way: Writing $L_1=:L_1(x,D)$, $Q_2=:Q_2(D)$ and applying L_1Q_2 to $\exp(\lambda_1x_1+\dots+\lambda_nx_n)$, we obtain by a straightforward calculation

$$L_1(x,\lambda)\cdot Q_2(\lambda)=0 \quad \text{for any} \quad x \in \Omega, \quad \lambda=(\lambda_1, \dots, \lambda_n) \in \mathbf{C}^n.$$

(Here, λ^α is defined by $\lambda_1^{\alpha_1}\dots\lambda_n^{\alpha_n}$.) For fixed x, $L_1(x,\lambda)$ and $Q_2(\lambda)$ are polynomials in λ, and since $Q_2 \neq 0$, we get $L_1(x,\lambda)\equiv 0$, which means that the operator L_1 vanishes. Therefore, Q_3 commutes with L, hence by Corollary 1, $P_1^{c_1}\dots P_l^{c_l}$ commutes with L. The remaining assertion of part a) is now obvious.

b) The proof of part b) coincides with the preceding one except for using Corollary 2 instead of Corollary 1.

c) Putting $P_j:=D_j$ in b), we conclude that L commutes with every D_j, $1\leqq j \leqq l$. Now we show: If $D_iL=LD_i$ for some i, then each coefficient of L is independent of x_i. For simplicity, we put $i=1$. Then, write $L=L_1+L_2$, where L_2 consists of all the terms involving the highest power of D_1, D_1^s say $(s \geqq 0)$:

$$L_2 := \sum_{\substack{|\alpha|\leqq m \\ \alpha_1=s}} A_\alpha(x)\,D^\alpha.$$

We get

$$0 = D_1L-LD_1 = \underbrace{D_1L_1-L_1D_1}_{=:\tilde{L}} + \sum_{\substack{|\alpha|\leqq m \\ \alpha_1=s}} (D_1A_\alpha)(x)\cdot D^\alpha. \tag{18}$$

Since $\tilde{L}$ does not contain powers D_1^s, we conclude by comparing the coefficients in (18): $\tilde{L}=0$ and $D_1 A_\alpha=0$ for any $|\alpha|\leqq m$ satisfying $\alpha_1=s$. Thus, by induction, we get $D_1 A_\alpha=0$ for every $|\alpha|\leqq m$.

Theorem 3c) is a generalization of two results in [1] (see Theorems 4 and 5).

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DIRECT AND SUBDIRECT DECOMPOSITIONS OF UNIVERSAL ALGEBRAS WITH A BOOLEAN ORTHOGONALITY

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1. Introduction. In this paper decompositions of universal algebras with a Boolean orthogonality as subdirect, strongly irredundant subdirect and direct products are obtained. In particular, we prove the direct product theorem on which many of the results in [6] depend. The prototypical example of an algebra with a Boolean orthogonality is an associative ring with no proper nilpotent elements: the relation $\perp$ defined by $x\perp y$ if and only if $xy=0$ is a Boolean orthogonality and the factors in our subdirect decompositions are rings with no proper divisors of zero.

We refer to Grätzer's book [8] for the conventions and fundamental concepts of universal algebra. In particular, the set of elements of an algebra, which is denoted by a capital Gothic letter, will be denoted by the corresponding Latin letter. Whenever more than one algebra is under discussion it is assumed that all the algebras are of the same type. The direct product of a set of algebras $\{\mathfrak{A}_\lambda : \lambda\in\Lambda\}$ is denoted by $\Pi\mathfrak{A}_\lambda$, and e_μ denotes the projection of $\Pi\mathfrak{A}_\lambda$ onto $\mathfrak{A}_\mu$. It will be convenient to always regard a subdirect product as a subalgebra of the direct product. Every algebra is assumed to be finitary and to possess a distinguished element 0 which is a nullary operation.

Let $\mathfrak{A}$ be an algebra. If θ is a congruence on $\mathfrak{A}$, then ker $\theta=\{a\in A : a\equiv 0(\theta)\}$ is the *kernel* of θ. A subset J of A is an *ideal* if it is the kernel of some congruence on $\mathfrak{A}$. If J is an ideal, then $\theta(J)$ denotes the smallest congruence which has J as its kernel. If $\mathfrak{A}$ and $\mathfrak{A}_1$ are algebras and $f:\mathfrak{A}\to\mathfrak{A}_1$ is a homomorphism, then $\ker f=\{a\in A : f(a)=0\}$. Notice that $\ker f$ is an ideal.

The general theory of Boolean orthogonalities has been developed by the first author in [5]. For the convenience of the reader we give the basic definitions as applied to our situation.

Let $\perp$ be a binary relation on the set A of elements of an algebra $\mathfrak{A}$. For $X\subseteq A$, $X^\perp=\{a\in A : a\perp x \text{ for all } x\in X\}$; $X^{\perp\perp}$ denotes $(X^\perp)^\perp$, and $x^\perp$ denotes $\{x\}^\perp$ for each $x\in A$. The set $\mathscr{B}(\mathfrak{A})=\{Y : Y=X^\perp \text{ for some } X\subseteq A\}$ is called the set of *polars*. With this notation a binary relation $\perp$ is a *Boolean orthogonality* on $\mathfrak{A}$ if and only if the following are satisfied:

(A$_1$) $x\perp 0$,
(A$_2$) $x\perp y$ implies $y\perp x$,
(A$_3$) $x\perp x$ implies $x=0$,
(A$_4$) $x^{\perp\perp}\cap y^{\perp\perp}=\{0\}$ implies $x\perp y$, for all $x, y\in A$, and
(A$_5$) every polar is an ideal.

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Conditions (A_1) through (A_4) ensure that $\mathscr{B}(\mathfrak{A})$ is a complete Boolean algebra. The infimum of a collection of polars $\{X_\lambda\}$ is their set-theoretic intersection $\bigcap X_\lambda$, their supremum is given by $\bigvee X_\lambda = (\bigcup X_\lambda)^{\perp\perp} = (\bigcap X_\lambda^\perp)^\perp$, and the complement of a polar X is $X^\perp$.

Boolean orthogonalities will be indiscriminately denoted by $\perp$.

2. Orthosubdirect products. The proof of the following proposition is straightforward and is omitted.

PROPOSITION 2.1. *Let $\{\mathfrak{A}_\lambda: \lambda \in \Lambda\}$ be a set of algebras. The relation $\perp$, defined on ΠA_λ by $x \perp y$ if and only if $e_\lambda(x)=0$ or $e_\lambda(y)=0$ for each $\lambda \in \Lambda$, is a Boolean orthogonality and $\mathscr{B}(\Pi\mathfrak{A}_\lambda) = \prod\limits_\lambda 2$ where 2 denotes the two element Boolean algebra.*

Let $\mathfrak{A}$ be an algebra with a Boolean orthogonality and $\{\mathfrak{A}_\lambda: \lambda \in \Lambda\}$ be a set of algebras. The algebra $\mathfrak{A}$ is an *orthosubdirect product* (respectively, *orthodirect product*) of $\{\mathfrak{A}_\lambda : \lambda \in \Lambda\}$ if $\mathfrak{A}$ is a subdirect product of $\{\mathfrak{A}_\lambda : \lambda \in \Lambda\}$ (respectively, $\mathfrak{A} = \Pi\mathfrak{A}_\lambda$) and the orthogonality on $\mathfrak{A}$ is the restriction to A of the orthogonality on $\Pi\mathfrak{A}_\lambda$. In other words, a subdirect product is an orthosubdirect product if and only if, for all $x, y \in A$, $x \perp y$ precisely when $e_\lambda(x)=0$ or $e_\lambda(y)=0$ for all $\lambda \in \Lambda$.

The following lemma is based on an idea from [7].

LEMMA 2.2. *Let $\mathfrak{A}$ be an algebra with a Boolean orthogonality and denote the set of prime ideals of $\mathscr{B}(\mathfrak{A})$ by Λ. For each $\lambda \in \Lambda$, define $P_\lambda = \bigcup\{Q: Q \in \lambda\}$. Then, for each $\lambda \in \Lambda$,*

(a) P_λ is an ideal of $\mathfrak{A}$,

(b) if F is finite and $F \subseteqq P_\lambda$, then $F^{\perp\perp} \subseteqq P_\lambda$,

(c) if $x \perp y$, then $x \in P_\lambda$ or $y \in P_\lambda$.

Also, $\bigcap P_\lambda = \{0\}$.

PROOF. Let $\lambda \in \Lambda$. Since λ is an ideal of $\mathscr{B}(\mathfrak{A})$, $\{Q: Q \in \lambda\}$ is a directed set of ideals of $\mathfrak{A}$ and hence P_λ is an ideal of $\mathfrak{A}$.

Suppose F is a finite subset of P_λ. Then $F \subseteqq Q \in \lambda$ for some Q and so $F^{\perp\perp} \subseteqq Q$. Since λ is an ideal of $\mathscr{B}(\mathfrak{A})$, $F^{\perp\perp} \in \lambda$ and hence $F^{\perp\perp} \subseteqq P_\lambda$.

If $x \perp y$, then $x^{\perp\perp} \cap y^{\perp\perp} = (0)$ so $x^{\perp\perp} \in \lambda$ or $y^{\perp\perp} \in \lambda$ because λ is a prime ideal. Thus, $x \in P_\lambda$ or $y \in P_\lambda$.

Finally, if $x \in \bigcap P_\lambda$, then $x^{\perp\perp} \in \bigcap\{\lambda: \lambda \in \Lambda\} = \{(0)\}$ and so $x=0$.

For algebras $\mathfrak{A}$ with the property that $\bigcap P_\lambda = (0)$ implies $\bigcap \theta(P_\lambda) = \omega$, the above lemma can be used to show that $\mathfrak{A}$ is an orthosubdirect product of $\{\mathfrak{A}/\theta(P_\lambda): \lambda \in \Lambda\}$. In particular, we obtain the following result.

THEOREM 2.3. *A weakly regular algebra $\mathfrak{A}$ is an orthosubdirect product of $\{\mathfrak{A}/\theta(P_\lambda): \lambda \in \Lambda\}$ where Λ is the set of prime ideals of $\mathscr{B}(\mathfrak{A})$ and for each $\lambda \in \Lambda$, $P_\lambda = \bigcup\{Q: Q \in \lambda\}$.*

PROOF. Let $\mathfrak{A}$ be a weakly regular algebra; that is, the map $\theta \to \ker\theta$ is a lattice isomorphism from the lattice of congruence to the lattice of ideals. Then, since $\bigcap P_\lambda = \{0\}$, $\bigcap \theta(P_\lambda) = \omega$. It follows that $\mathfrak{A}$ is a subdirect product of $\{\mathfrak{A}/\theta(P_\lambda): \lambda \in \Lambda\}$.

Suppose $x \perp y$. Then Lemma 2.2 (c) implies that $e_\lambda(x)=0$ or $e_\lambda(y)=0$ for each $\lambda \in \Lambda$. Conversely, if $e_\lambda(x)=0$ or $e_\lambda(y)=0$ for each $\lambda \in \Lambda$, then $x \in P_\lambda$ or $y \in P_\lambda$ for each $\lambda \in \Lambda$. From Lemma 2.2 (b) we see that $x^{\perp\perp} \cap y^{\perp\perp} \subseteqq P_\lambda$ for all $\lambda \in \Lambda$ and so $x^{\perp\perp} \cap y^{\perp\perp} = \{0\}$; that is, $x \perp y$.

There are algebras with Boolean orthogonalities which can not be represented as orthosubdirect products. The relation $\perp$ defined by $x \perp y$ if and only if $xy=0$ is a Boolean orthogonality on the 5-element commutative semiring S discussed in [4]. This semiring is subdirectly irreducible and $\mathscr{B}(S)$ is the 4-element Boolean algebra.

Theorem 2.3 would be more satisfactory if the (trivial) Boolean orthogonality on the factors $\mathfrak{A}/\theta(P_\lambda)$ had a closer algebraic relation to the Boolean orthogonality on $\mathfrak{A}$. More precisely, we would like to have that $x^{\perp\perp} \cap y^{\perp\perp} \subseteqq P_\lambda$ implies that $x \in P_\lambda$ or $y \in P_\lambda$. This is the case when the Boolean orthogonality satisfies the following finiteness condition:

for any two elements $x, y \in A$ there is a finite set $F \subseteqq A$ such that $x^{\perp\perp} \cap y^{\perp\perp} = F^{\perp\perp}$

(in the teminology of [5], $\mathscr{D}(\mathfrak{A})$ is a lattice). If A is an associative ring which is *reduced* (has no nonzero nilpotent elements), then the relation $\perp$ defined by $x \perp y$ if and only if $xy=0$ is a Boolean orthogonality and the finiteness condition is satisfied because $x^{\perp\perp} \cap y^{\perp\perp} = \{xy\}^{\perp\perp}$. In this case Theorem 2.3 yields that A is a subdirect product of rings without proper divisors of zero. The formula $x^{\perp\perp} \cap y^{\perp\perp} = \{xy\}^{\perp\perp}$ is valid in a rather general setting [5; Theorem 3.5] and in view of this Theorem 2.3 generalizes the subdirect decomposition theorem for reduced semigroups of ADAMS [2] and CORNISH [4] as well as the decomposition theorem for reduced alternative rings given in [9] and [13].

The remainder of this paper is concerned with decompositions of algebras $\mathfrak{A}$ with a Boolean orthogonality such that $\bigcap\{\theta(M): M \text{ is maximal in } \mathscr{B}(\mathfrak{A})\} = \omega$. For many algebras, including all weakly regular algebras, this condition is equivalent to the condition that the Boolean algebra $\mathscr{B}(\mathfrak{A})$ is atomic.

PROPOSITION 2.4. *Let $\mathfrak{A}$ be an algebra with a Boolean orthogonality and let $\{M_\lambda: \lambda \in \Lambda\}$ be the set of maximal polars.*

(a) If $\bigcap\{\theta(M_\lambda): \lambda \in \Lambda\} = \omega$, then $\bigcap\{M_\lambda: \lambda \in \Lambda\} = \{0\}$ and so $\mathscr{B}(\mathfrak{A})$ is atomic.

(b) If $\mathscr{B}(\mathfrak{A})$ is atomic, then $\bigcap\{M_\lambda: \lambda \in \Lambda\} = \{0\}$ and, for any $\mu \in \Lambda$, $M_\mu^\perp = \bigcap\{M_\lambda: \lambda \in \Lambda, \lambda \neq \mu\}$.

We omit the routine verification of Proposition 2.4.

PROPOSITION 2.5. *Let $\mathfrak{A}$ be an algebra with a Boolean orthogonality $\perp$ and $\mathfrak{C}$ a subalgebra of $\mathfrak{A}$ such that for all $\{0\} \neq P \in \mathscr{B}(\mathfrak{A})$, $P \cap C \neq \{0\}$. Then the restriction of $\perp$ to C is a Boolean orthogonality on $\mathfrak{C}$, $\mathscr{B}(\mathfrak{C}) = \{P \cap C: P \in \mathscr{B}(\mathfrak{A})\}$, and the maps $\mu: \mathscr{B}(\mathfrak{C}) \to \mathscr{B}(\mathfrak{A})$ and $\nu: \mathscr{B}(\mathfrak{A}) \to \mathscr{B}(\mathfrak{C})$ defined by $\mu(Q) = Q^{\perp\perp}$ and $\nu(P) = P \cap C$ are mutually inverse Boolean isomorphisms.*

PROOF. We begin by showing that if $X \subseteqq A$, then $X^\perp = (X^{\perp\perp} \cap C)^\perp$. Because $(X^{\perp\perp} \cap C) \cap (X^{\perp\perp} \cap C)^\perp = \{0\}$, our assumption about C forces $X^{\perp\perp} \cap (X^{\perp\perp} \cap C)^\perp = \{0\}$. Hence, $(X^{\perp\perp} \cap C)^\perp \subseteqq X^\perp$. Also, $X^{\perp\perp} \cap C \subseteqq X^{\perp\perp}$ implies that $X^\perp \subseteqq (X^{\perp\perp} \cap C)^\perp$ so that $X^\perp = (X^{\perp\perp} \cap C)^\perp$.

Clearly, $\perp'$, the restriction of $\perp$ to C, satisfies conditions $(A_1)-(A_3)$. Suppose $x, y \in C$. Then

$$x^{\perp'\perp'} \cap y^{\perp'\perp'} = \{(x^\perp \cap C)^\perp \cap C\} \cap \{(y^\perp \cap C)^\perp \cap C\} =$$

$$= \{(x^\perp)^\perp \cap C\} \cap \{(y^\perp)^\perp \cap C\} = x^{\perp\perp} \cap y^{\perp\perp} \cap C$$

and so $x^{\perp'\perp'}\cap y^{\perp'\perp'}=\{0\}$ implies $x^{\perp\perp}\cap y^{\perp\perp}\cap C=\{0\}$. This in turn implies that $x^{\perp\perp}\cap y^{\perp\perp}=\{0\}$, so $\perp'$ satisfies (A_4).

If $X\subseteq C$, then $X^{\perp'}=X^{\perp}\cap C$ and if $X\subseteq A$, $X^{\perp}\cap C=(X^{\perp\perp}\cap C)^{\perp}\cap C=(X^{\perp\perp}\cap C)^{\perp'}$. Thus, to establish ($A_5$) and the equality $\mathscr{B}(\mathfrak{C})=\{P\cap C\colon\ P\in\mathscr{B}(\mathfrak{A})\}$ it suffices to observe that $P\cap C$ is an ideal of $\mathfrak{C}$ for each $P\in\mathscr{B}(\mathfrak{A})$.

If $Q=P\cap C\in\mathscr{B}(\mathfrak{C})$, then $\nu\mu(Q)=\nu\big((P\cap C)^{\perp\perp}\big)=\nu\big((P^{\perp})^{\perp}\big)=\nu(P)=P\cap C=Q$. Also, if $P\in\mathscr{B}(\mathfrak{A})$, $\mu\nu(P)=\mu(P\cap C)=(P\cap C)^{\perp\perp}=(P^{\perp})^{\perp}=P$. Thus μ and ν are mutually inverse and since they are clearly order preserving, they are Boolean isomorphisms.

A subdirect product $\mathfrak{A}$ of a set $\{\mathfrak{A}_\lambda\colon\ \lambda\in\Lambda\}$ of algebras is *strongly irredundant* if, for each $\mu\in\Lambda$, $e_{(\mu)}\colon\ \mathfrak{A}\to\prod\limits_{\lambda\neq\mu}\mathfrak{A}_\lambda$, defined by $e_\lambda(e_{(\mu)}(a))=e_\lambda(a)$ for each $a\in A$ and each $\lambda\neq\mu$, has $\ker e_{(\mu)}\neq\{0\}$.

THEOREM 2.6. *An algebra* $\mathfrak{A}$ *with a Boolean orthogonality is a strongly irredundant orthosubdirect product if and only if* $\cap\{\theta(M)\colon M$ *is maximal in* $\mathscr{B}(\mathfrak{A})\}=\omega$. *In this case* $\mathfrak{A}$ *is a strongly irredundant orthosubdirect product of* $\{\mathfrak{A}/\theta_M : M$ *is maximal in* $\mathscr{B}(\mathfrak{A})\}$ *where* $\{\theta_M : M$ *is maximal in* $\mathscr{B}(\mathfrak{A})\}$ *is any set of congruences of* $\mathfrak{A}$ *such that* $\cap\theta_M=\omega$ *and* $\ker\theta_M=M$ *for all maximal polars* M.

PROOF. Suppose that $\mathfrak{A}$ is a strongly irredundant orthosubdirect product of $\{\mathfrak{A}_\lambda : \lambda\in\Lambda\}$. Let $\{0\}\neq P\in\mathscr{B}(\Pi\mathfrak{A}_\lambda)$. Because of the way the orthogonality is defined on ΠA_λ, $\ker e_{(\mu)}\subseteq P$ for some $\mu\in\Lambda$. Thus, since the subdirect product is strongly irredundant, $P\cap A\neq\{0\}$. It now follows from Propositions 2.1 and 2.5 that the set of maximal polars in $\mathscr{B}(\mathfrak{A})$ is $\{\ker e_\lambda : \lambda\in\Lambda\}$. Since $\mathfrak{A}$ is a subdirect product of $\{\mathfrak{A}_\lambda : \lambda\in\Lambda\}$, $\cap\theta_\lambda=\omega$ where $x\equiv y(\theta_\lambda)$ if and only if $e_\lambda(x)=e_\lambda(y)$. Clearly $\theta(\ker e_\lambda)\subseteq\theta_\lambda$ for all $\lambda\in\Lambda$, so $\cap\theta(\ker e_\lambda)=\omega$.

Conversely, suppose that $\{\theta_M : M$ is maximal in $\mathscr{B}(\mathfrak{A})\}$ is a set of congruences of $\mathfrak{A}$ such that $\cap\theta_M=\omega$ and $\ker\theta_M=M$ for all maximal polars M. Then $\mathfrak{A}$ is a subdirect product of $\{\mathfrak{A}/\theta_M : M$ is a maximal in $\mathscr{B}(\mathfrak{A})\}$.

If $x\perp y$, then $x^{\perp\perp}\cap y^{\perp\perp}=\{0\}$ and so $x^{\perp\perp}\subseteq M$ or $y^{\perp\perp}\subseteq M$ for each maximal polar M. Thus, $x\equiv 0(\theta_M)$ or $y\equiv 0(\theta_M)$ for each maximal polar M. Now suppose that, for each maximal polar M, either $x\equiv 0(\theta_M)$ or $y\equiv 0(\theta_M)$. Then $x^{\perp\perp}\cap y^{\perp\perp}\subseteq\cap\{M\colon M$ is maximal in $\mathscr{B}(\mathfrak{A})\}=\{0\}$. Thus, $x\perp y$ and so $\mathfrak{A}$ is an orthosubdirect product of $\{\mathfrak{A}/\theta_M : M$ is maximal in $\mathscr{B}(\mathfrak{A})\}$. Finally, Proposition 2.4 (b) implies that this subdirect product is strongly irredundant.

Let $\mathfrak{A}$ be a strongly irredundant orthosubdirect product of $\{\mathfrak{A}_\lambda\colon\ \lambda\in\Lambda\}$. The above proof shows that $\ker e_\lambda$ is a maximal polar and it follows that $x^{\perp\perp}\cap y^{\perp\perp}\subseteq\ker e_\lambda$ implies that $x\in\ker e_\lambda$ or $y\in\ker e_\lambda$, for each $y\in\Lambda$. Thus, the desired algebraic relation holds between the Boolean orthogonality on $\mathfrak{A}$ and the "trivial" orthogonality on the factors $\mathfrak{A}_\lambda$.

Now suppose that A is a semiprime ring. The relation $\perp$ defined by $x\perp y$ if and only if $xAy=\{0\}$ is a Boolean orthogonality, the polars are just the annihilator ideals and $x^{\perp\perp}\cap y^{\perp\perp}=(xAy)^{\perp\perp}$ for all $x, y\in A$. If every annihilator ideal of A, except A itself, is contained in a maximal annihilator ideal, then $\mathscr{B}(A)$ is atomic and so A is a strongly irredundant orthosubdirect product of a set of factor rings $\{A/I_\lambda : \lambda\in\Lambda\}$. Each of the rings A/I_λ is prime because $xAy\subseteq I_\lambda$ implies $x^{\perp\perp}\cap y^{\perp\perp}=(xAy)^{\perp\perp}\subseteq I_\lambda$ which implies that $x\in I_\lambda$ or $y\in I_\lambda$. Thus we obtain a theorem of LEVY [10, 3.2]: an associative ring A is a strongly irredundant subdirect product of prime rings if and only if A is semiprime and every annihilator ideal of A, except A itself, is contained in a maximal annihilator ideal.

The following two concepts are required for the next theorem.

A subset S of elements of an algebra $\mathfrak{A}$ with a Boolean orthogonality is *orthogonal* if $s\perp t$ for all $s, t\in S$, $s\neq t$.

Any set of congruences $\{\psi_\lambda : \lambda\in\Lambda\}$ on an algebra $\mathfrak{A}$ defines a relation $\leqq_\psi$ on A as follows: $x\leqq_\psi y$ if and only if $x\in\ker\psi_\lambda$ or $x\equiv y(\psi_\lambda)$ for each $\lambda\in\Lambda$. It is straightforward to check that $\leqq_\psi$ is a partial order when $\cap\{\psi_\lambda :\lambda\in\Lambda\}=\omega$.

THEOREM 2.7. *An algebra* $\mathfrak{A}$ *with a Boolean orthogonality is an orthodirect product if and only if there is a set of congruences* $\{\theta_M\colon\ M$ *is maximal in* $\mathscr{B}(\mathfrak{A})\}$ *such that* $\cap\theta_M=\omega$, $\ker\theta_N=N$ *and* $(\bigcap\limits_{M\neq N}\theta_M)\circ\theta_N=\iota$ *for each maximal polar* N *and every orthogonal subset of* A *has a least upper bound with respect to the partial order* $\leqq_\theta$.

PROOF. Suppose that $\mathfrak{A}$ is an orthodirect product of $\{\mathfrak{A}_\lambda : \lambda\in\Lambda\}$. As in the proof of Theorem 2.6, $\{\ker e_\lambda : \lambda\in\Lambda\}$ is the set of maximal polars of $\mathfrak{A}$. For each $\lambda\in\Lambda$ define θ_λ by $x\equiv y(\theta_\lambda)$ if and only if $e_\lambda(x)=e_\lambda(y)$. Then $\cap\theta_\lambda=\omega$ and for each $\mu\in\Lambda$, $\ker\theta_\mu=\ker e_\mu$ and $(\bigcap\limits_{\lambda\neq\mu}\theta_\lambda)\circ\theta_\mu=\iota$ [12, Proposition 1.4, page 66]. Let $K=\{a_\beta :\beta\in B\}$ be an orthogonal subset of A. For each $\lambda\in\Lambda$, either $e_\lambda(a_\beta)=0$ for all $\beta\in B$ or there is a unique $\beta=\beta(\lambda)$ such that $e_\lambda(a_{\beta(\lambda)})\neq 0$. Define $s\in A$ by $e_\lambda(s)=0$ if $e_\lambda(a_\beta)=0$ for all $\beta\in B$ and $e_\lambda(s)=e_\lambda(a_{\beta(\lambda)})$ otherwise. The element s is well-defined and is clearly the least upper bound of K with respect to $\leqq_\theta$.

Conversely, suppose that $\{\theta_M : M$ is maximal in $\mathscr{B}(\mathfrak{A})\}$ is a set of congruences satisfying the conditions given in the statement of the theorem. Since $\theta(M)\leqq\theta_M$ for all maximal polars M, $\cap\theta(M)=\omega$. Thus, by Theorem 2.6, $\mathfrak{A}$ is a strongly irredundant orthosubdirect product of $\{\mathfrak{A}_M=\mathfrak{A}/\theta_M : M$ is maximal in $\mathscr{B}(\mathfrak{A})\}$. Moreover, if $b\in\Pi A_M$ and $a\in A$ satisfy $\{M\colon\ e_M(b)\neq e_M(a)\}$ is finite, then $b\in A$ [12, Proposition 1.5, page 67].

Let $x\in\Pi A_M$. For each maximal polar N define $x^N\in\Pi A_M$ by $e_N(x^N)=e_N(x)$ and $e_M(x^N)=0$ if $M\neq N$. Since $\{M\colon\ e_M(x^N)\neq e_M(0)\}$ is finite, $x^N\in A$. Also $\{x^N\colon N$ is a maximal polar$\}$ is an orthogonal set. Let $s\in A$ be the least upper bound of this set with respect to $\leqq_\theta$. If $x^N\neq 0$, then $x^N\notin\ker\theta_N$ and so $x^N\equiv s(\theta_N)$ because $x^N\leqq_\theta s$. Thus, $e_N(x)=e_N(x^N)=e_N(s)$ for all maximal polars N such that $x^N\neq 0$. Now suppose that $x^N=0$. Define $b\in\Pi A_M$ by $e_N(b)=0$ and $e_M(b)=e_M(s)$ for all $M\neq N$. Since $\{M\colon\ e_M(b)\neq e_M(s)\}$ is finite, $b\in A$. Clearly $x^M\leqq_\theta b$ for all maximal polars M and so $s\leqq_\theta b$. It follows that $e_N(s)=0$ and so $x=s\in A$. Thus $A=\Pi A_M$ and the proof is complete.

When $\mathfrak{A}$ is a weakly regular algebra there is only one choice for the set of congruences $\{\theta_M : M$ is a maximal polar$\}$, so in this case the partial order is denoted simply by $\leqq$. Moreover, for weakly regular algebras with a Boolean orthogonality the partial order $\leqq$ may be defined intrinsically [6, Proposition 2.1]. For instance, if $\mathfrak{A}$ is a multioperator group, then $x\leqq y$ if and only if $x\perp y-x$. The results of ABIAN [1], CHACRON [3], and MYUNG and JIMENEZ [11] can be deduced from Theorem 2.7. These and other applications of 2.7 are discussed in [6].

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PROOF OF JACKSON'S THEOREM FOR DIFFERENTIABLE FUNCTIONS. II

By

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1. It was the purpose of Kis—Vértesi [2] to give a proof of Timan's approximation theorem for continuous functions by an interpolatory algebraic polynomial constructed on the basis of an identity proved by A. H. Turecki [4]. One question, however remained: How does the derivative of the polynomial behave comparing with the derivative of the function or what about the approximation of differentiable functions by these polynomials? It is the sole aim of this paper to partially give answer to the above problem by somewhat modifying the definition of the polynomial.

2. To begin with, we, in brief, describe the actual form of the polynomial. Let $-1 \leqq x \leqq 1$, $x = \cos t$

$$x_{kn} = \cos t_{kn} \tag{2.1}$$

$$t_{kn} = \frac{2k\pi}{2n+1}, \quad k = 0, \ldots, n. \tag{2.2}$$

(We shall write k instead of kn for simplicity.)

Then for f belonging to $C^{(s)}[-1,1] = \{f \in C[-1,1] : f^{(s)} \in C[-1,1]\}$, we define the polynomial

$$L_{ns}(f, x) = \sum_{k=0}^{n} \sum_{r=0}^{s} (x-x_k)^r f^{(r)}(x_k)\, v_k(x) \quad (s=0, 1), \tag{2.3}$$

where

$$\begin{cases} v_0(x) = u_0(t), \\ v_k(x) = u_k(t) + u_{-k}(t), \quad k = 1, \ldots, n, \end{cases} \tag{2.4}$$

$$u_k(t) = 4l_k^3(t) - 3l_k^4(t), \tag{2.5}$$

$$l_k(t) = \frac{\sin \frac{2n+1}{2}(t-t_k)}{(2n+1)\sin \frac{1}{2}(t-t_k)}, \quad k = -n, \ldots, n. \tag{2.6}$$

For $s=0$, our polynomial is the same as that of Vértesi—Kis [2]. For $s=1$, we shall prove the following.

Theorem. *Let $f \in C^{(1)}[-1,1]$. Then for any natural numbér n, we have for $-1 \leqq x \leqq 1$*

$$|L_{n1}(f, x) - f(x)| \leqq \frac{C_1}{n}\, \omega_{f'}\left(\frac{1}{n}\right) \tag{A}$$

$$|L'_{n1}(f, x) - f(x)| \leqq C_2\, \omega_{f'}\left(\frac{1}{n}\right), \tag{B}$$

here $\omega_{f'}(\cdot)$ denotes the modulus of continuity of f'.

It is remarkable that the estimates for the derivative of the polynomial in the case of $s=0$ can be easily obtained.

3. For the proof of the theorem we require a lemma which is essentially contained in [2]. The technique of the proof is similar to that of [2].

LEMMA. *For* $0\leqq t\leqq\pi$,

$$\text{(3.1)}\qquad \begin{cases} \text{i)} & |l_k^3(t)+l_{k+1}^3(t)|\leqq \dfrac{3\pi}{\sigma_k^4}, \\ \text{ii)} & \left|\dfrac{d}{dt}\left(l_k^3(t)+l_{k+1}^3(t)\right)\right| \leqq \dfrac{9\pi(2n+1)}{\sigma_k^4}, \end{cases}$$

where $\sigma_k=(2n+1)\sin \frac{1}{2}(t-t_k)$.

PROOF OF THE THEOREM. We have from the identity $\sum_{k=-n}^{n} u_k(t)=1$ (see [2]),

$$\text{(3.2)}\qquad f(x)-L_{n1}(f,x)=\sum_{k=0}^{n}[f(x)-f(x_k)-(x-x_k)f'(x_k)]\,v_k(x)=\sum_{k=-n}^{n} g_k u_k(t)$$

where

$$\text{(3.3)}\qquad g_k=f(\cos t)-f(\cos t_k)-(\cos t-\cos t_k)f'(\cos t_k).$$

We define j by

$$\text{(3.4)}\qquad |t-t_j|\leqq\frac{\pi}{2n+1}.$$

Since the functions involved are 2π-periodic, the increase or decrease of the numbers k in multiples of $2n+1$ does not affect them so we can write

$$f(x)-L_{n1}(f,x)=\sum_{k=j-n}^{j+n} g_k u_k(t).$$

Now break the sum into four parts

$$\text{(3.5)}\qquad f(x)-L_{n1}(f,x)=g_j u_j(t)+4\sum_{k=j-n}^{j-1} g_k l_k^3(t)+$$

$$+4\sum_{k=j+1}^{j+n} g_k l_k^3(t)-3\sum_{\substack{k=j-n\\k\neq j}}^{j+n} g_k l_k^4(t)=S_1+S_2+S_3+S_4.$$

We will frequently use

$$\text{(3.6)}\qquad f(u)-f(v)=(u-v)f'(v)+O\big(|u-v|\,\omega_{f'}(|u-v|)\big)$$

which is easy to establish.

Using (3.6), (3.3) and the estimates for $u_k(t)$ we can show that

$$\text{(3.7)}\qquad |S_1|\leqq C_1\left(\frac{\sin t}{n}+\frac{1}{n^2}\right)\omega_{f'}\left(\frac{\sin t}{n}+\frac{1}{n^2}\right).$$

Similarly if we make use of

$$\text{(3.8)}\qquad |l_k(t)|\leqq\frac{1}{|\sigma_k|},\quad k\neq j,$$

we obtain

$$|l_k(t)|\leqq\frac{1}{2i-1},\quad (k\neq j,\ i=|k-j|).$$

So we find that

$$\text{(3.9)}\qquad |S_4|\leqq 3\sum_{\substack{k=j-n\\k\neq j}}^{j+n}|g_k|\,l_k^4(t)\leqq C_2\sum|\cos t-\cos t_k|\,\omega_{f'}(|\cos t-\cos t_k|)\,l_k^4(t)\leqq$$

$$\leqq C_2\omega_{f'}\left(\frac{1}{n}\right)\sum\{n(\cos t-\cos t_k)^2+|\cos t-\cos t_k|\}\frac{1}{\sigma_k^4}\leqq$$

$$\leqq C_2\omega_{f'}\left(\frac{1}{n}\right)\sum\left[\frac{4n}{(2n+1)^2}\frac{1}{\sigma_k^2}+\frac{2}{2n+1}\frac{1}{|\sigma_k^3|}\right]\leqq C_2\frac{1}{n}\omega_{f'}\left(\frac{1}{n}\right)\sum\frac{1}{\sigma_k^2}.$$

For S_2' and S_3, the tools used in [2] are well-applicable. In short, for the sake of completeness, we have,

$$S_2=4\sum_{k=j-n}^{j-1} g_k l_k^3(t)=4\sum\{g_k[l_k^3(t)+l_{k+1}^3(t)]+(g_{k+1}-g_k)\,l_{k+1}^3\}.$$

Let

$$\text{(3.10)}\qquad S_2=S_2'+S_2''.$$

For S_2' using the lemma and (3.6) we have at once

$$\text{(3.11)}\qquad |S_2'|\leqq C_5\frac{1}{n}\omega_{f'}\left(\frac{1}{n}\right).$$

Now,

$$\text{(3.12)}\qquad |g_{k+1}-g_k|\leqq|\cos t_k-\cos t_{k+1}|\,\omega_{f'}(|\cos t_k-\cos t_{k+1}|)+$$

$$+|\cos t-\cos t_{k+1}|\,\omega_{f'}(|\cos t_k-\cos t_{k+1}|).$$

Combining (3.12), (3.9) and the estimates for $(\cos t_k-\cos t_{k+1})$ and $(\cos t-\cos r_{k+1})$, we easily obtain

$$\text{(3.13)}\qquad |S_2''|\leqq C_6\frac{1}{n}\omega_{f'}\left(\frac{1}{n}\right).$$

Taking (3.13), (3.11), (3.9) and (3.7) together, we have (A).

To prove the second part, let us write, using the identity (3.3)

$$\text{(3.14)}\qquad L_{n1}'(f;x)-f'(x)=\sum_{k=0}^{n}[f(x_k)-f(x)+$$

$$+(x-x_k)f'(x_k)]\,v_k'(x)+\sum_{k=0}^{n}\big(f'(x_k)-f'(x)\big)\,v_k(x).$$

The estimate for the second sum is analogous to [2].

For the first sum, using the method of the previous proof we obtain,

$$I_1 = -\sum_{k=0}^{n} [f(x)-f(x_k)-(x-x_k)f'(x_k)]\, v_k'(x) = \tag{3.15}$$

$$= \frac{1}{\sin t} \sum_{k=j-n}^{j+n} g_k \left[\frac{d}{dt}(u_k(t))\right] = \frac{1}{\sin t} g_j \frac{d}{dt}(u_j(t)) + \frac{4}{\sin t} \sum_{k=j-n}^{j-1} g_k \frac{d}{dt}(l_k^3) +$$

$$+\frac{4}{\sin t} \sum_{k=j+1}^{j+n} g_k \frac{d}{dt}(l_k^3) - \frac{3}{\sin t} \sum_{\substack{k=j-n \\ k\neq j}}^{j+n} g_k \frac{d}{dt}(l_k^4) = I_{11}+I_{12}+I_{13}+I_{14}.$$

The estimates for I_{11} and I_{14} are analogous if we keep in mind that $|\sin nt| \leqq n|\sin t|$. To estimate I_{12} and I_{13} using here again the above methods, we can write

$$I_{12} = \frac{4}{\sin t} \sum_{k=j-n}^{j-1} g_k \frac{d}{dt}(l_k^3 + l_{k+1}^3) +$$

$$+\frac{4}{\sin t} \sum_{k=j-n}^{j-1} (g_{k+1}-g_k) \frac{d}{dt}(l_{k+1}^3) = I_{12}' + I_{12}''.$$

For I_{12}' making use of the second part of the lemma, we have

$$|I_{12}'| \leqq C_{11}\, \omega_{f'}\left(\frac{1}{n}\right).$$

In a similar way we can obtain estimates for I_{12}'' if we use (3.13) and the estimates for other quantities in suitable form.

Thus we have

$$|L_{n1}'(f, x) - f'(x)| \leqq C_2\, \omega_{f'}\left(\frac{1}{n}\right).$$

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SUBGROUPS OF SEMIFREE GROUPS

By

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1. Introduction

We call a group semifree iff it has a presentation with defining relations $[a, b]=1$ between the defining generators only. In [1] it is proved:

Theorem 1.1. *Every Abelian subgroup of a semifree group is free-Abelian.*

Our main result in this paper is

Theorem 1.2. *Let u and v be elements of a semifree group. Then $[u, v]\neq 1$ implies that $\{u, v\}$ is a basis of a free group.*

To prove Theorem 1.2 cancellation arguments as firstly developed by Nielsen are used.

Corollary 1.3. *Every subgroup of a semifree group generated by two elements is semifree.*

Furthermore, it will be shown that Corollary 1.3 is not true for subgroups generated by three elements. The counterexample group that is a subgroup of a semifree group has the property that every Abelian subgroup is free-Abelian.

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2. Basic facts on semifree groups

The results stated in this section are proved in [1].

Let A be a set. A *word* over A is a finite sequence $a_1^{\alpha_1}a_2^{\alpha_2}\dots a_n^{\alpha_n}$, where $a_i\in A$ and $\alpha_i\neq 0$ are integers. Let e be the *empty word*. $a_1^{\alpha_1}\dots a_n^{\alpha_n}$ is *reduced* iff $a_i\neq a_{i+1}$.

If R is a set of words, we use (A, R) to denote the group F/N, where F is the free group with basis A and N is the normal closure of R in F. Then a group is *semifree* iff it has a presentation (A, R) where $R\subseteqq \{[a, b]: a, b\in A\}$.

We use elements of A (letters) and words over A to denote elements of (A, R). Obviously, for every $u\in(A, R)$ there is some reduced word $a_1^{\alpha_1}\dots a_n^{\alpha_n}$ in the coset of u. Two words u and v represent the same element of the semifree group (A, R) iff we can carry u into v applying a finite number of the following transformations:

(R1) Replace $a_1^{\alpha_1}\dots a_i^{\alpha_i}\, a_{i+1}^{\alpha_{i+1}}\dots a_k^{\alpha_k}$ by $a_1^{\alpha_1}\dots a_i^{\alpha_i}aa^{-1}\, a_{i+1}^{\alpha_{i+1}}\dots a_k^{\alpha_k}$ for some $a\in A$.

(R2) If $a_i=a=a_{i+1}$, replace $a_i^{\alpha_i}a_{i+1}^{\alpha_{i+1}}$ in $a_1^{\alpha_1}\dots a_k^{\alpha_k}$ by $a^{\alpha_i+\alpha_{i+1}}$ and delete it if $\alpha_i+\alpha_{i+1}=0$.

(R3) Replace $a_1^{\alpha_1}\dots a_i^{\alpha_i}a_{i+1}^{\alpha_{i+1}}\dots a_k^{\alpha_k}$ by $a_1^{\alpha_1}\dots a_{i+1}^{\alpha_{i+1}}\, a_i^{\alpha_i}\dots a_k^{\alpha_k}$ if $[a_i, a_{i+1}]\in R$.

$u=a_1^{\alpha_1}\dots a_n^{\alpha_n}$ is a *minimal form* for u iff n is minimal. We call n the *length* $\lambda(u)$ of u. Minimal forms are reduced.

LEMMA 2.1. *If $u=a_1^{\alpha_1}\dots a_n^{\alpha_n}$ and $u=b_1^{\beta_1}\dots b_m^{\beta_m}$ is a minimal form of u, then $m\leqq n$ and it is possible to get $b_1^{\beta_1}\dots b_m^{\beta_m}$ from $a_1^{\alpha_1}\dots a_n^{\alpha_n}$ applying* (R2) *and* (R3) *only.*

Lemma 2.1 immediately implies

LEMMA 2.2. *Let $a_1^{\alpha_1}\dots a_n^{\alpha_n}$ and $b_1^{\beta_1}\dots b_m^{\beta_m}$ be minimal forms of u. Then*
(i) *$n=m$ and it is possible to transform one minimal form into the other using* (R3) *only.*
(ii) $\{a\in A\colon a=a_i \text{ for some } i\}=\{a\in A\colon a=b_i \text{ for some } i\}$.

By Lemma 2.2 (ii) it is possible to speak about *the* letters of u. Let $a_1^{\alpha_1}\dots a_n^{\alpha_n}$ be a minimal form of u and a be a letter of u. By Lemma 2.2 (i) the following definitions are correct:

The *a-sequence* of u is the sequence $a^{\alpha_{i_1}}\dots a^{\alpha_{i_m}}$ where $a^{\alpha_{i_j}}=a_{i_j}^{\alpha_{i_j}}$ is the j-th a-power in $a_1^{\alpha_1}\dots a_n^{\alpha_n}$.

If $[a, b]\neq 1$, we can similarly define the $\{a, b\}$-sequence $c_1^{\gamma_1}\dots c_m^{\gamma_m}$ of u where $c_j^{\gamma_j}$ is the j-th occurrence of a power of a or b in $a_1^{\alpha_1}\dots a_n^{\alpha_n}$.

Furthermore, if $[a, b]\neq 1$ and $[b, c]\neq 1$ and u has a minimal form $\dots a^{\alpha}\dots b^{\beta}\dots c^{\gamma}\dots$, it is possible to say "b^{β} lies between a^{α} and c^{γ}", "b^{β} is on the right of a^{α}", and so on, since by Lemma 2.2 (i) this is true in every minimal form of u.

We make the following convention: $u=w_1w_2\dots w_n$ is a minimal form means that the w_i's are minimal forms and the concatenation of the w_i's is a minimal form of u. We suppose that $w_i\neq e$ if there is no other assumption.

a is a *first (last) letter* of power α of u iff u has a minimal form $a^{\alpha}u'$ (resp. $u'a^{\alpha}$).

u is cyclic reduced iff u has no minimal form $a^{\alpha}u'a^{\beta}$ with $\alpha<0<\beta$ or $\beta<0<\alpha$.

We are now interested in the cancellation of some letter a in products uv.

LEMMA 2.3. (i) *Semifree groups are torsion free.*
(ii) *Let a be a first (last) letter of power α of u, and let ε be any natural >0. If there is some b with $[a, b]\neq 1$ in u, then a is a first (last) letter of power α of u^{ε}. Otherwise a is a first (last) letter of power $\alpha\cdot\varepsilon$ of u^{ε}. u and u^{ε} contain the same letters.*

If $a^{\alpha_1}\dots a^{\alpha_n}$ is the a-sequence of u, $a^{\beta_1}\dots a^{\beta_m}$ is the a-sequence of v, and $a^{\alpha_1}\dots a^{\alpha_n}a^{\beta_1}\dots a^{\beta_m}$ or $a^{\alpha_1}\dots a^{\alpha_n+\beta_1}\dots a^{\beta_m}$ is the a-sequence of uv, then we say that there is no full cancellation of an a-power between u and v.

LEMMA 2.4. *If $a^{\alpha_1}\dots a^{\alpha_n}$ is the a-sequence of u and $a^{\beta_1}\dots a^{\beta_m}$ is the a-sequence of v and $\alpha_n+\beta_1\neq 0$, then there is no full cancellation of an a-power between u and v.*

If B is any subset of a group G, then $\mathrm{Gp}(B)$ is used to denote the subgroup of G generated by B. The following lemma is easliy proved.

LEMMA 2.5. *If (A, R) is semifree and $B\subseteq A$, then $\mathrm{Gp}(B)=(B, R_B)$, where $R_B=R\cap\{[a, b]\colon a, b\in B\}$.*

Let W be any subset of a semifree group (A, R) and C be the set of letters that occur in some element of W. W is called to be *connected* iff for every $a, b\in C$ there are $c_1, \dots, c_n\in C$ such that $[a, c_1]\neq 1$, $[c_i, c_{i+1}]\neq 1$ for $1\leqq i<n$, and $[c_n, b]\neq 1$. Then W is connected iff the graph $\langle C, \varrho\rangle$ is connected where $\varrho(a, b)$ iff $[a, b]\neq 1$ for $a, b\in C$.

W is *strongly connected* iff wWw^{-1} is connected for every w that contains letters of C only.

LEMMA 2.6. *If (A, R) is semifree and $A=\bigcup_{1\leqq i\leqq k} A_i$ with $[a_i, a_j]=1$ for $a_i\in A_i$, $a_j\in A_j$, $i\neq j$, then $(A, R)=\bigoplus_{1\leqq i\leqq k}\mathrm{Gp}(A_i)$.*

3. Nielsen transformations

Let G be any group. An elementary Nielsen transformation works on vectors $(u_1, u_2, \dots)$ where $u_i\in G$. It is one of the following transformations:

(T1) Replace some u_i by u_i^{-1}.
(T2) Replace some u_i by u_iu_j where $j\neq i$.
(T3) Delete some u_i where $u_i=1$.

A finite product of such transformations is a *Nielsen transformation;* it is *regular* if there is no factor of type (T3), and *singular* otherwise. (The definitions above are taken from [3].)

LEMMA 3.1. *Every regular Nielsen transformation has an inverse.*

LEMMA 3.2. *If $(u_1, \dots, u_n)$ is carried in $(v_1, \dots, v_m)$ by a Nielsen transformation, then $u_1, \dots, u_n$ and $v_1, \dots, v_m$ generate the same subgroup.*

LEMMA 3.3. *If $(u_1, \dots, u_n)$ is a basis of a free subgroup F of G and it is carried in $(v_1, \dots, v_m)$ by a regular Nielsen transformation, then $n=m$ and $(v_1, \dots, v_m)$ is a basis of F.*

LEMMA 3.4. *u and v generate a cyclic subgroup of G iff it is possible to apply a singular Nielsen transformation on (u, v).*

To check freeness we use the following well-known

LEMMA 3.5. *Let $u_1, \dots, u_n$ be elements of G. Then $(u_1, \dots, u_n)$ is a basis for a free subgroup of G iff $w(u_1, \dots, u_n)\neq 1$ for every reduced work $w(x_1, \dots, x_n)$.*

4. The main proof

THEOREM 4.1. *Let u and v be elements of a semifree group $G=(A, R)$ such that $\{u, v\}$ is strongly connected. If u and v do not generate a cyclic subgroup, then (u, v) is a basis of a free subgroup.*

By Lemma 2.5 we can assume

(1) The letters occurring in u or v are exactly the elements of A.

By a transformation on (u, v) a finite product of elementary Nielsen transformations and inner automorphisms is meant. Such a transformation is regular iff there is no factor of type (T3), and singular otherwise. Since u and v do not generate a cyclic subgroup of G, Lemma 3.4 implies

(2) All possible transformations of (u, v) are regular.

If σ is a regular transformation on (u, v), then $\mathrm{Gp}(u, v)$ is free with basis (u, v) iff $\mathrm{Gp}(\sigma(u), \sigma(v))$ is free with basis $(\sigma(u), \sigma(v))$.

This follows from Lemmas 3.1 and 3.3. Using Lemma 3.5 for a suitable transformation σ it will be shown that $\mathrm{Gp}(\sigma(u), \sigma(v))$ is free with basis $(\sigma(u), \sigma(v))$. By (2) σ is regular and, therefore, as stated above, $\mathrm{Gp}(u, v)$ is free with basis (u, v).

Remark that regular Nielsen transformations and inner automorphisms "commute". Therefore, if $\{u, v\}$ is strongly connected, then by (1) and (2) $\{\sigma(u), \sigma(v)\}$ is strongly connected for every regular transformation σ. Using a suitable regular transformation, (1), (2), and Lemma 2.5 again, we can suppose

(3) For every transformation σ, A is the set of letters occurring in $\sigma(u)$ or $\sigma(v)$. A is connected.

Case 1. There exist some letter $a \in A$ and a transformation σ such that $\sigma(u) = a^\alpha u' a^\beta$ and $\sigma(v) = a^\gamma v' a^\delta$ are minimal forms, where u' and v' contain letters that do not commute with a, $\beta+\gamma \neq 0$, $\alpha+\delta \neq 0$, $\beta-\delta \neq 0$, $\gamma-\alpha \neq 0$, and the exponents $\alpha, \beta, \gamma, \delta$ need not differ from 0.

By Lemmas 3.5, 2.3, and 2.4 $\mathrm{Gp}(\sigma(u), \sigma(v))$ is free with basis $(\sigma(u), \sigma(v))$.

Case 2. There exist some transformation σ and some letter a such that $\sigma(u) = a^\alpha u'$, a is not a letter of u' and $\sigma(v)$, and every letter of u' commutes with a (resp. u and v are exchanged). Assume that $\sigma(u) = a^\alpha u'$. By (3) there is some letter d of $\sigma(v)$ with $[a, d] \neq 1$. By assumption d is not a letter of $\sigma(u)$. By Lemma 2.3 (ii) every $\sigma(u)^\varepsilon$ contains a and not d and every $\sigma(v)^\varepsilon$ contains d and not a. If $w(x_1, x_2)$ is any nontrivial reduced word then the $\{a, d\}$-sequence of $w(\sigma(u), \sigma(v))$ is the concatenation of the a-sequences of the $\sigma(u)$-powers in w and the d-sequences of the $\sigma(v)$-powers in w. Lemma 3.5 implies the assertion.

Case 3. Not Case 1, not Case 2, and there is some letter $a \in A$ and a transformation σ such that a is not a letter of both $\sigma(u)$ and $\sigma(v)$.

We show by induction on $\lambda(\sigma(u)) + \lambda(\sigma(v))$:

(4) If a and σ fulfil the condition of Case 3, then there is some w such that a is not a letter of both $w\sigma(u)w^{-1}$ and $w\sigma(v)w^{-1}$, and $w\sigma(u)w^{-1}$ and $w\sigma(v)w^{-1}$ are cyclic reduced.

Assume that $\sigma(u)$ and $\sigma(v)$ are not both cyclic reduced and a is an element of $\sigma(u)$. We consider the more difficult case when $\sigma(u)$ has a minimal form $c^\alpha u' c^{-\beta}$ with $\alpha, \beta > 0$ or $\alpha, \beta < 0$. Since the condition of Case 1 is not fulfilled $\sigma(v)$ has w. l. o. g. a minimal form $c^\alpha v'$ or $c^\gamma v''$, where c commutes with every letter of v'' ($\gamma = 0$ is possible). First we suppose $\sigma(v) = c^\alpha v'$. Then $c \neq a$, because c is an element of $\sigma(v)$. This implies that a is not a letter of $c^{-\alpha}\sigma(v)c^\alpha$. Furthermore

$$\lambda(c^{-\alpha}\sigma(u)\,c^\alpha) + \lambda(c^{-\alpha}\sigma(v)\,c^\alpha) = \lambda(u' c^{\alpha-\beta}) + \lambda(v' c^\alpha) <$$

$$< \lambda(c^\alpha u' c^{-\beta}) + \lambda(c^\alpha v') = \lambda(\sigma(u)) + \lambda(\sigma(v)).$$

The assertion follows from the induction hypothesis. If $\sigma(v) = c^\gamma v''$, where c commutes with every letter of v'', then $\gamma \neq 0$ implies $c \neq a$. Therefore $c^{-\alpha}\sigma(v)c^\alpha$ does not contain a for every γ. We have

$$\lambda(c^{-\alpha}\sigma(u)\,c^\alpha) + \lambda(c^{-\alpha}\sigma(v)\,c^\alpha) = \lambda(u' c^{\alpha-\beta}) + \lambda(c^\gamma v'') <$$

$$< \lambda(c^\alpha u' c^{-\beta}) + \lambda(c^\gamma v'') = \lambda(\sigma(u)) + \lambda(\sigma(v)).$$

The assertion follows as above from the induction hypothesis.

The situation of Case 3 implies furthermore:

(5) There are letters a and b in A such that $[a, b] \neq 1$ and either a is not in $\sigma(v)$ and b is in $\sigma(v)$ or a is not in $\sigma(u)$ and b is in $\sigma(u)$.

To prove (5) assume that Case 3 is given and a is an element of $\sigma(u)$. We get a and b with (5) since the following procedure must break off. Suppose that the letters $a_0 = a, a_1, \dots, a_k$ are chosen such that

(5′) $\{a_0, \dots, a_k\}$ is connected, every a_i is an element of $\sigma(u)$ but not an element of $\sigma(v)$, and every a_i commutes with all elements of $\sigma(v)$.

Since $\{\sigma(u), \sigma(v)\}$ is connected, there exist some c in $\sigma(u)$ or in $\sigma(v)$ and some j with $0 \leqq j \leqq k$ and $[a_j, c] \neq 1$. By (5′) c is not an element of $\sigma(v)$. Then either c and some letter of $\sigma(v)$ fulfil (5) or $a_0, \dots, a_k, c$ satisfy (5′). Since $\{\sigma(u), \sigma(v)\}$ is connected, the letters of $\sigma(u)$ cannot satisfy (5′), therefore the procedure breaks off, and we get some a, b with (5).

Applying at first (4) and then (5), and using u instead of $w\sigma(u)w^{-1}$ and v instead of $w\sigma(v)w^{-1}$ for convenience, we can suppose w. l. o. g.

(6) There is some letter a in u that is not a letter of v and some letter b in v such that $[a, b] \neq 1$. u and v are cyclic reduced.

To verify the condition of Lemma 3.5 we show the following:

(7) For every nontrivial reduced word $w(x_1, x_2)$ the a-sequence of $w(u, v)$ is the concatenation of the a-sequences of u in $w(u, v)$.

Firstly, we consider the case that $w(x_1, x_2)$ contains positive (negative, resp.) powers of x_1 only. For $w(x_1, x_2) = x_1^n$ (7) is true since u is cyclic reduced and by the conditions "not Case 1" and "not Case 2", a cannot be a first and a last letter of u at the same time. If (7) were false, there would be some subword $u^\gamma v^\delta u^\varepsilon$ so that the last a-power of u^γ can "touch" the first a-power of u^ε after some applications of (R2) and (R3) (Lemma 2.1). Since (7) is fulfilled for x_1^n we can suppose $\gamma = \varepsilon = 1$ (-1, resp.). If there is one a-power in u only, let $u = u_1 a^\alpha u_3$ be a minimal form of u. Then for every letter c of $u_3 v^\delta u_1$, $[a, c] = 1$, by assumption. We can replace (u, v) by $(u_1^{-1} u v^\delta u_1, u_1^{-1} v u_1) = (a^\alpha(u_3 v^\delta u_1), u_1^{-1} v u_1)$ using a suitable transformation. Bur $\sigma(u) = a^\alpha(u_3 v^\delta u_1)$ and $\sigma(v) = u_1^{-1} v u_1$ fulfil the condition of Case 2, a contradiction.

Otherwise, there is some minimal form $u_1 a^\alpha u_2 a^\beta u_3$ of u, where a does not occur in u_1 and u_3. As above $[a, c] = 1$ for every letter c of $u_3 v^\delta u_1$. There is some transformation τ such that $\tau(u) = u_1^{-1} u v^\delta u_1 = a^\alpha u_2 (u_3 v^\delta u_1) a^\beta$ and $\tau(v) = u_1^{-1} v u_1$. a does not occur in $u_1^{-1} v u_1 \cdot u_1^{-1} v u_1$ contains b of (6) by Lemma 4.2 below, and in $u_2(u_3 v^\delta u_1)$

there is some d with $[a, d]\neq 1$, since such a d is in u_2, and $[a, c]=1$ for every c of $u_3 v^\delta u_1$. Therefore we have Case 1, a contradiction.

Now let $w(x_1, x_2)$ be any nontrivial reduced word. We find a minimal form $w_1(x_1, x_2)x_2^{\beta_1}w_2(x_1, x_2)x_2^{\beta_2}\dots x_2^{\beta_{l-1}}w_1(x_1, x_2)$ of $w(x_1, x_2)$ where:

(i) x_1 is a last letter of $w_i(x_1, x_2)$ for $1\leqq i<l$.

(ii) x_1 is a first letter of $w_i(x_1, x_2)$ for $1<i\leqq l$.

(iii) There are either only positive or only negative powers of x_1 in $w_i(x_1, x_2)$ for $1\leqq i\leqq l$.

(iv) If the powers of x_1 are positive in w_i, then the powers of x_1 in w_{i+1} are negative $(1\leqq i<l)$. If the powers of x_1 are negative in w_i, then the powers of x_1 are positive in w_{i+1} $(1\leqq i<l)$.

Let us consider $w_i(u, v)v^\beta w_{i+1}(u, v)$. Then $w_i(x_1, x_2)=w_i'(x_1, x_2)x_1^\gamma$, $w_{i+1}(x_1, x_2)=x_1^\delta w_{i+1}'(x_1, x_2)$ with $\lambda\big(w_j(x_1, x_2)\big)>\lambda\big(w_j'(x_1, x_2)\big)$ for $j=i, i+1$. Assume w. l. o. g. $\gamma>0>\delta$. ($\delta>0>\gamma$ is similar.) Let $u=u'a^\alpha r$ be a minimal form, where a is not a letter of r, and every power in r occurs in every minimal form of u on the right of a^α. Since (7) holds for $w_i(u, v)$ and $w_{i+1}(u, v)$, we get $w_i(u, v)=\overline{w}_i(u, v)a^\alpha r$, $w_{i+1}(u, v)=$ $=r^{-1}a^{-\alpha}\overline{w}_{i+1}(u, v)$. Now it is suffcient to show that $rv^\beta r^{-1}$ contains the letter b of assumption (6). This follows from

LEMMA 4.2. *Assume that the letter b occurs in a word y, and y is cyclic reduced. Then b is a letter of ryr^{-1} for every r.*

PROOF. The lemma will be proved by induction on $\lambda(r)$. If there is any cancellation of a full power of b in ryr^{-1}, assume w. l. o. g. that $y=sb^\beta y'$ and $r=r'b^{-\beta}s^{-1}$ are minimal forms, where s does not contain b and $s=e$ is possible. Then $ryr^{-1}=$ $=r'(y'sb^\beta)r'^{-1}$. Since y is cyclic reduced, there is no cancellation in $(y'sb^\beta)$, b is a letter of $(y'sb^\beta)$, and $(y'sb^\beta)$ is cyclic reduced. Since $\lambda(r')<\lambda(r)$, by induction hypothesis $ryr^{-1}=r'(y'sb^\beta)r'^{-2}$ contains b. O. E. D.

Case 4. Not Case 1 and for every transformation σ, each of $\sigma(u)$ and $\sigma(v)$ contains all letters of A.

Let a be any element of A. Define $\lambda_a(w)$ to be the length of the a-sequence of w. By the assumption above we have $\lambda_a\big(\sigma(u)\big)>0$ and $\lambda_a\big(\sigma(v)\big)>0$ for every transformation σ. Now we take σ in such a way that $\lambda_a\big(\sigma(u)\big)+\lambda_a\big(\sigma(v)\big)$ is minimal. For convenience we use u instead of $\sigma(u)$ and v instead of $\sigma(v)$. Therefore

(8) $\lambda_a(u)+\lambda_a(v)$ is minimal with respect to transformations.

We need

(9) $\lambda_a(xx)=2\lambda_a(x)$ for every $x\in\{u, v, u^{-1}, v^{-1}\}$.

To prove (9) assume w. l. o. g. $x=u$. Firstly consider $\lambda_a(x)=1$ and $\lambda_a(xx)<2\lambda_a(x)$. There is some minimal form $u_1a^\alpha zu_3$ of x, where every letter of u_3 is on the right of a^α in every minimal form of u, every letter of u_1 is on the left of a^α in every minimal form of u, and every letter of z commutes with a. Then $\lambda_a(uu)=1$ implies $u_1^{-1}=u_3$. $u_1^{-1}uu_1=a^\alpha z$ does not contain every letter of A, a contradiction to the assumption of Case 4.

Now suppose $\lambda_a(u)\geqq 2$ and $\lambda_a(uu)<2\lambda_a(u)$. Similarly as above you get a minimal form $u_1a^\alpha u_2a^\beta u_1^{-1}$ of u. Then $u_1^{-1}uu_1=a^\alpha u_2a^\beta$. Since we are not in Case 1 and every letter of A occurs in $u_1^{-1}vu_1$, w.l.o.g. $a^\alpha v'$ is a minimal form of $u_1^{-1}vu_1$. It follows

$$\lambda_a(a^{-\alpha}u_1^{-1}uu_1a^\alpha)+\lambda_a(a^{-\alpha}u_1^{-1}vu_1a^\alpha) = \lambda_a(u_2a^{\alpha+\beta})+\lambda_a(v'a^\alpha)<\lambda_a(u)+\lambda_a(v),$$

a contradiction to (8).

Let us assume that there is at most consolidation but not full cancellation of a-powers. That means:

(10) For every $x, y\in\{u, u^{-1}, v, v^{-1}\}$ with $x\neq y^{-1}$ either

(i) $\lambda_a(xy)=\lambda_a(x)+\lambda_a(y)$ or

(ii) $x=w_1a^\alpha w_2$, $y=w_1a^\beta w_3$ are minimal forms, where a is not a letter of w_1 and $\alpha-\beta\neq 0$.

(10) and $\lambda_a(u)$, $\lambda_a(v)>1$ imply that $w(u, v)\neq 1$ for any nontrivial reduced $w(x_1, x_2)$.

It remains to settle the case when (w.l.o.g.) a^α is the only a-power in u. If for every $x, y\in\{u, v, u^{-1}\}$ (10) (i) is true, then there is nothing to do. Otherwise, by (9) w.l.o.g. u and v satisfy (10) (ii).

Applying $w_1^{-1}\dots w_1$, w.l.o.g. $u=a^\alpha w_2$, $v=a^\beta w_3$ are minimal forms of u and v. Then a is not a last letter of u, since otherwise $[a, c]=1$ for every letter c of u and by the assumption of Case 4 for every $c\in A$. This would be a contradiction because A is connected. Similarly, a is not a last letter of v if $\lambda_a(v)=1$. If $\lambda_a(v)>1$, then a is not a last letter of v, too. Otherwise, we would have minimal forms $u=a^\alpha w_2$, $v=a^\beta w_3'a^\gamma$, where by (10) $\alpha, \beta, \gamma\neq 0$, $\alpha+\gamma\neq 0$, $\alpha-\beta\neq 0$. Then the condition of Case 1 is fulfilled, a contradiction.

Therefore, $\lambda_a(uv)=\lambda_a(vu)=\lambda_a(u)+\lambda_a(v)$. If $\lambda_a(uv^{-1})=\lambda_a(u)+\lambda_a(v)$, the assertion follows. Otherwise, $\lambda_a(uv^{-1})=\lambda_a(v)$ and we get a contradiction in the following way:

If $\lambda_a(v)=1$, then $uv^{-1}=a^{\alpha-\beta}z$, where every letter of z commutes with a. If σ is the transformation with $\sigma(u)=u$, $\sigma(v)=uv^{-1}$, the assumption of Case 4 is violated, a contradiction.

If $\lambda_a(v)>1$, then we have a minimal form $uv^{-1}=a^\gamma w'a^{-\beta}$. By $\lambda_a(uv^{-1})=\lambda_a(v)$, $\gamma\neq 0$. If $\alpha\neq\gamma$, the conditions of Case 1 are fulfilled for u and uv^{-1} (remember that $\alpha\neq\beta$). If $\alpha=\gamma$, then

$$\lambda_a\big(a^{-\alpha}(uv^{-1})\,a^\alpha\big)+\lambda_a(a^{-\alpha}ua^\alpha) = \lambda_a(w'a^{\alpha-\beta})+\lambda_a(w_2a^\alpha)<$$

$$< \lambda_a(uv^{-1})+\lambda_a(u) = \lambda_a(v)+\lambda_a(u).$$

This contradicts (8).

Contrary to (10) it remains to suppose that there are $x, y\in\{u, v, u^{-1}, v^{-1}\}$, $x\neq y^{-1}$ with minimal forms $x=w_1a^\alpha w_2$, $y=w_1a^\alpha w_3$, where w_1 does not contain the letter a and $w_1=e$ is possible. By (9), $x\neq y$. Applying $w_1^{-1}\dots w_1$ we can assume w.l.o.g.

(11) $u=a^\alpha u'$ and $v=a^\alpha v'$, where a is not a first letter of u', v'.

Then $\lambda_a(u)$, $\lambda_a(v)>1$, because otherwise $\lambda_a(u^{-1}v)+\lambda_a(u)<\lambda_a(u)+\lambda_a(v)$ or $\lambda_a(u^{-1}v)+\lambda_a(v)<\lambda_a(u)+\lambda_a(v)$, a contradiction to (8).

We have $\lambda_a(uv)=\lambda_a(vu)=\lambda_a(u)+\lambda_a(v)$. By $\lambda_a(u)\geqq 2$, $\lambda_a(v)\geqq 2$ there are minimal forms $u=a^\alpha u_1a^\beta u_2$ and $v=a^\alpha v_1a^\gamma c_2$, where a is not a letter of u_2 and v_2, every power in u_2 occurs in every minimal form of u on the right of a^β, every power in v_2 occurs in every minimal form of v on the right of a^γ, and $u_2, v_2\neq e$.

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Assume w.l.o.g. $\lambda_a(u) \geqq \lambda_a(v)$. By (8), $\lambda_a(u^{-1}v)$, $\lambda_a(v^{-1}u) \geqq \lambda_a(u) \geqq 2$.

If $u_2 \neq v_2$ or $\beta \neq \gamma$, then full cancellation of a-powers is only possible in products $u^{-1}v$. (Remark that $u_2^{-1}v_2 \neq 1$ implies the existence of some letter d in $u_2^{-1}v_2$ with $[a, d] \neq 1$ by the construction of u_2 and v_2.)

$u^{-1}v$ has a minimal form $u_2^{-1}a^{-\beta}za^{\gamma}v_2^{-1}$ since $\lambda_a(u^{-1}v) \geqq \lambda_a(u) \geqq 2$. We can write every nontrivial word $w(u, v)$ as a product $w_1 w_2 \dots w_n$ with $w_l = u^{-1}v$, or $w_l = v^{-1}u$, or $w_l \in \{u, v, u^{-1}, v^{-1}\}$ and $w_l = u$ implies $w_{l-1} \neq v^{-1}$, $w_l = v$ implies $w_{l-1} \neq u^{-1}$, $w_l = u^{-1}$ implies $w_{l+1} \neq v$, and $w_l = v^{-1}$ implies $w_{l+1} \neq u$. Then there is no full cancellation of an a-power between w_l and w_{l+1}. It follows $w(u, v) \neq e$. If $u_2 = v_2$ and $\beta = \gamma$, we prove

(12) $\lambda_a(uv^{-1}) > \lambda_a(u)$ and $\lambda_a(u^{-1}v) > \lambda_a(u)$.

By (8) uv^{-1} has a normal form $a^{\alpha}za^{-\alpha}$. Then $\lambda_a(uv^{-1}) \leqq \lambda_a(u)$ would imply

$$\lambda_a\big(a^{-\alpha}(uv^{-1})\,a^{\alpha}\big) + \lambda_a(a^{-\alpha}va^{\alpha}) < \lambda_a(uv^{-1}) + \lambda_a(v) \leqq \lambda_a(u) + \lambda_a(v),$$

contrary to (8).

$u^{-1}v$ has a minimal form $u^{-1}v = u_2^{-1}a^{-\beta}za^{\beta}u_2$ by (8), $u_2 = v_2$ and $\beta = \gamma$. Then $\lambda_a(u^{-1}v) \leqq \lambda_a(u)$ would imply

$$\lambda_a(a^{\beta}u_2u^{-1}vu_2^{-1}a^{-\beta}) + \lambda_a(a^{\beta}u_2vu_2^{-1}a^{-\beta}) < \lambda_a(u^{-1}v) + \lambda_a(v) \leqq \lambda_a(u) + \lambda_a(v),$$

a contradiction to (8).

(12) implies

(13) $\lambda_a(xy) > \max\big(\lambda_a(x), \lambda_a(y)\big)$ for every $x, y \in \{u, v, u^{-1}, v^{-1}\}$ with $x \neq y^{-1}$.

If z has a minimal form $r_0a^{\alpha_1}r_1a^{\alpha_2}r_2 \dots r_{l-1}a^{\alpha_l}r_l$, where $(a^{\alpha_1}, a^{\alpha_2}, \dots, a^{\alpha_l})$ is the a-sequence of z and $r_0 = e$, $r_l = e$ is possible, define $m(z)$ to be the subword $a^{\alpha_{k+1}}$ if $l = 2k+1$ and $m(z) = a^{\alpha_k}r_ka^{\alpha_{k+1}}$ if $l = 2k$.

Let $w = z_1^{\varepsilon_1} \dots z_n^{\varepsilon_n}$ with $z_l \in \{u, v\}$, $\varepsilon_l \in \{1, -1\}$ be any nontrivial reduced word. Then by (13) the subwords $m(z_l)$ will not be cancelled in w. That means if $m(z_l) = a^{\alpha_{k+1}}$, $m(z_l)$ remains in w, if $m(z_l) = a^{\alpha_k}r_ka^{\alpha_{k+1}}$, only consolidation of a^{α_k} and $a^{\alpha_{k+1}}$ is possible. Hence $w \neq 1$. Q.E.D.

5. Consequences

THEOREM 1.2. *Let u, v be elements of a semifree group (A, R). Then $[u, v] \neq 1$ implies that $\{u, v\}$ is a basis of a free subgroup of rank 2.*

PROOF. Applying a suitable inner automorphism we can suppose:

$(*)$ For every w that contains letters of u and v only the same letters as in u and v occur in wuw^{-1} and wvw^{-1}.

By Lemma 2.5 it is possible to assume that A is the set of letters in u and v. If A is connected, then by $(*)$ $\{u, v\}$ is strongly connected and the assertion follows from Theorem 4.1.

Otherwise, $A = \bigcup_{1 \leqq l \leqq k} A_l$ with $k > 1$, every A_l is connected, and $[a_l, a_j] = 1$ for $a_l \in A_l$ and $a_j \in A_j$ with $i \neq j$. Then $(A, R) = \bigoplus_{1 \leqq l \leqq k} \mathrm{Gp}\,(A_l)$ by Lemma 2.6. Let $u = u_1u_2 \dots u_k$ and $v = v_1v_2 \dots v_k$ be minimal forms with $u_i, v_i \in \mathrm{Gp}\,(A_i)$. Since $[u, v] \neq 1$, there must be some j with $[u_j, v_j] \neq 1$. Since A is the set of all letters in u and v, by Lemma 2.2 (ii) A_j is the set of all letters of u_j and v_j. Therefore, $\{u_j, v_j\}$ is strongly connected by $(*)$. Applying Theorem 4.1 the assertion follows. Q.E.D.

Further consequences of Theorem 4.1 are the results of [1].

THEOREM 5.1. *Let u, v be elements of a semifree group (A, R). Then $[u, v] = 1$ iff there are elements w, w_i, and integers α_i, β_i $(1 \leqq i \leqq n)$ such that:*

(i) *If $i \neq j$, then every letter of w_i commutes with every letter of w_j.*
(ii) *Every w_i is connected.*
(iii) $u = w_1 \prod_{1 \leqq i \leqq n} w_i^{\alpha_i} w^{-1}$ *and* $v = w \prod_{1 \leqq i \leqq n} w_i^{\beta_i} w^{-1}$.

THEOREM 1.1. *Every Abelian subgroup of a semifree group is free-Abelian.*

6. Counterexamples

Unfortunately, it is not possible to sharpen Theorem 1.2. We need the following result of Baumslag:

THEOREM 6.1 (BAUMSLAG [2]). *For all elements u, v, w of a free group it holds: $[u, v] = w^n \neq 1$ with $n \geqq 1$ implies $n = 1$.*

THEOREM 6.2 *Let G be the semifree group*

$$(\{a, b, c\},\quad \{[a, c]\}) \oplus (\{x, y, z\},\quad \{[y, z]\})$$

and G_1 be $\mathrm{Gp}\,(\{ax, by, cz\})$. *Then G_1 is not semifree.*

PROOF. Since $[ax, cz] = [x, z] \neq 1$ and $[by, cz] = [b, c] \neq 1$, G_1 is not Abelian and therefore not cyclic. Furthermore, $[x, z], [b, c] \in G_1$, $\big[[x, z], [b, c]\big] = 1$, but $[x, z]$, $[b, c]$ do not generate a cyclic subgroup of G. Then $\mathrm{Gp}\,(\{[x, z], [b, c]\})$ is not a cyclic subgroup of G_1 and therefore G_1 is not free. By Theorem 1.2, it is not possible to generate G_1 by fewer than three elements.

Assume that G_1 is semifree. The facts above imply $G_1 = (\{u_1, u_2, u_3\}, R)$, where R contains one or two $[u_i, u_j]$. Then w.l.o.g. either

(i) $G_1 = \mathrm{Gp}(\{u_1\}) \oplus (\mathrm{Gp}(\{u_2\}) * \mathrm{Gp}(\{u_3\}))$

or

(ii) $G_1 = \mathrm{Gp}(\{u_1\}) * (\mathrm{Gp}(\{u_2\}) \oplus \mathrm{Gp}(\{u_3\}))$.

Case (i). Let $ax = r_1s_1$, $by = r_2s_2$, $cz = r_3s_3$ with $r_i \in \mathrm{Gp}\,(\{u_1\})$ and $s_i \in \mathrm{Gp}\,(\{u_2\}) * \mathrm{Gp}\,(\{u_3\})$ for $1 \leqq i \leqq 3$. Since ax, by, cz do not pairwise commute, $s_i \neq 1$ for $1 \leqq i \leqq 3$. As a subgroup of the free group $\mathrm{Gp}\,(\{u_2\}) * \mathrm{Gp}\,(\{u_3\})$, $G_2 = \mathrm{Gp}\,(\{s_1, s_2, s_3\})$ is a free subgroup of G. Then $\big[[s_1, s_3], [s_2, s_3]\big] = 1$ implies the existence of some $v \in G_2$ and some integers n, m with $[s_1, s_3] = v^m$ and $[s_2, s_3] = v^n$. By the Theorem 6.1 of Baumslag $|m| = 1$ and $|n| = 1$. If $[s_1, s_3] = [s_2, s_3]$, then $[s_3, s_1s_2^{-1}] = 1$ and therefore $[cz, ax(by)^{-1}] = 1$, a contradiction by the definition of G. If $[s_1, s_3] = [s_2, s_3]^{-1}$, then $[x, y] = [ax, cz] = [s_1, s_3] = [s_2, s_3]^{-1} = [by, cz]^{-1} = [b, c]^{-1}$, a contradiction. Therefore, the only possibility of G_1 to be semifree is

Case (ii). Consider $[[x, z], [b, c]]=1$ and $[x, z], [b, c]\in G_1$. By Theorem 5.1 there are v, w and intergers n, m, n_2, m_2, n_3, m_3 such that $[x, z]=wv^n w^{-1}$ and $[b, c]=wv^m w^{-1}$, where u_1 is a letter of v or $[x, z]=wu_2^{n_2}u_3^{n_3}w^{-1}$ and $[b, c]=wu_2^{m_2}u_3^{m_3}w^{-1}$. W.l.o.g. we can suppose $w=e$. Otherwise, apply the inner automorphism $w^{-1}\dots w$ of G_1.

The first case above is impossible because otherwise v would be a nontrivial common element of the subgroups Gp $(\{x, y, z\})$ and Gp $(\{a, b, c\})$ of G.

If $[x, z]=u_2^{n_2}u_3^{n_3}$ and $[b, c]=u_2^{m_2}u_3^{m_3}$, assume that $u_2=r_2s_2$ and $u_3=r_3s_3$ with $r_i\in$ Gp $(\{x, y, z\})$ and $s_i\in$ Gp $(\{a, b, c\})$. Since $[x, z]\in$ Gp $(\{x, y, z\})$, it follows $s_2^{n_2}s_3^{n_3}=e$ and $[x, z]=r_2^{n_2}r_3^{n_3}$. By $[u_2, u_3]=1$ we have $[r_2, r_3]=1$. Therefore and by Theorem 1.1 Gp $(\{r_2, r_3\})$ is free-Abelian of rank at most two. If there is some r with $r_2=r^i$ and $r_3=r^j$, then $[x, z]=r^{in_2+jn_3}$. By Lemma 2.3 r is an element of the free group Gp $(\{x,z\})$. By the Theorem of Baumslag (6.1) $[x, z]=r$ or $[x, z]=r^{-1}$. Therefore, Gp $(\{[x, z]\})=$ Gp $(\{r_2, r_3\})$. Otherwise, (r_2, r_3) is a basis of free-Abelian group. Since $[b, c]=s_2^{m_2}s_3^{m_3}$ and $r_2^{m_2}r_3^{m_3}=1$, it follows $m_2=m_3=0$. But this contradicts $[b, c]=s_2^{m_2}s_3^{m_3}\neq 1$.

Analogously, we can show that Gp $(\{[b, c]\})=$ Gp $(\{s_2, s_3\})$. Therefore, Gp $(\{u_2\})\oplus$ Gp $(\{u_3\})\subseteqq [G_1, G_1]$. It follows that $G_1/[G_1, G_1]$ is a cyclic group. This contradicts the fact that rank $(G_1/[G_1, G_1])=3$. Q.E.D.

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ON THE DISCREPANCY OF SEQUENCES (αn^σ)

By
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If (x_n) is a sequence of real numbers in $[0, 1)$ and C_A denotes the characteristic function of the set A, we denote by $D_N^*((x_n))$ the term $\sup\limits_{x\in[0,1)}\left|\frac{1}{N}\sum\limits_{n=1}^{N} C_{[0,x)}(x_n)-x\right|$ and by $D_n(((x_n))$ the term $\sup\limits_{0\leqq\alpha<\beta\leqq 1}\left|\frac{1}{N}\sum\limits_{n=1}^{N} C_{[\alpha,\beta)}(x_n)-(\beta-\alpha)\right|$ and call it the discrepancy of the sequence (x_n).

It is well known and one can easily prove that $D_N^*((x_n))\leqq D_N((x_n))\leqq 2D_N^*((x_n))$.

Now let $f\colon [1,\infty)\to\mathbf{R}$ be differentiable with monotone derivative. Let $\lim\limits_{t\to\infty} t^{-1}f(t)=\lim\limits_{t\to\infty}(tf'(t))^{-1}=0$. Then it is known that if $(x_n)=(f(n))$ mod 1 $D_N((x_n))=O(N^{-1}f(N)+(Nf'(N))^{-1})$ as $N\to\infty$ (see [2] or [3]). Furthermore $\varliminf\limits_{N\to\infty} Nf'(N)D_N((x_n))>0$ (see [2]).

From this we deduce, if $f(x)=\alpha x^\sigma$ and $0<\sigma\leqq\frac{1}{2}$, $\alpha\neq 0$, then $D_N((x_n))=O(N^{-\sigma})$ and this estimate cannot be sharpened.

We shall calculate the terms

$$\varlimsup_{N\to\infty} N^\sigma D_N^*((x_n)),\quad \varliminf_{N\to\infty} N^\sigma D_N^*((x_n))\quad\text{and}\quad \lim_{N\to\infty} N^\sigma D_N((x_n))$$

in the case

$$\alpha>0,\quad 0<\sigma<\frac{1}{2}\quad\text{or if}\quad \alpha>0,\quad \alpha^2\notin\mathbf{Q}\quad\text{and}\quad \sigma=\frac{1}{2}.$$

These calculations will be easy.

Furthermore we shall calculate the term $\varlimsup\limits_{N\to\infty} N^\sigma D_N^*((x_n))$ in the case $\alpha>0$, $\alpha^2=\frac{1}{q}$ for some $q\in\mathbf{N}$, and $\sigma=\frac{1}{2}$ and $\varliminf\limits_{N\to\infty} N^\sigma D_N^*((x_n))$ if $1\leqq q\leqq 4$. These calculations will be somewhat more complicated.

In the case $\frac{1}{2}<\sigma<1$ we can show that the best known estimation $D_N((x_n))=O(N^{\sigma-1})$ is *not* sharp.

1. An auxiliary result

Let $\{x\}$ be the fractional part of the real number x.

PROPOSITION 1. *Let $f\colon [1,\infty)\to\mathbf{R}$ be continuous and increasing in the strict sense. Let $x\in[0,1]$, $N\in\mathbf{N}$ and $f(1)+x\leqq f(N)$. Then*

$$\sum_{n=1}^{N} C_{[0,x)}(\{f(n)\}) = \sum_{f(1)\leqq k\leqq f(N)} \big([f^{-1}(k+x)]-[f^{-1}(k)]+C_{\mathbf{Z}}(f^{-1}(k))-$$

$$-C_{\mathbf{Z}}(f^{-1}(k+x))\big)+C_{(0,x)}(\{f(1)\})\big([f^{-1}([f(1)]+x)]-C_{\mathbf{Z}}(f^{-1}([f(1)]+x))\big)-$$

$$-C_{[0,x)}(\{f(N)\})\big(f^{-1}([f(N)]+x)]-N-C_{\mathbf{Z}}(f^{-1}([f(N)]+x))\big).$$

PROOF. Evidently

$$\sum_{n=1}^{N} C_{[0,x)}(\{f(N)\}) = \sum_{k\in\mathbf{Z}}\sum_{n=1}^{N} C_{[k,k+x)}(f(n)) =$$

$$=\sum_{k\in\mathbf{Z}}\sum_{n=1}^{N} C_{f^{-1}([k,k+x))}(n) = \sum_{f(1)-x<k\leqq f(N)} \operatorname{card}\big(\mathbf{Z}\cap[1,N]\cap f^{-1}([k,k+x))\big).$$

Now split this sum into three sums of the form $f(1)-x<k<f(1)$, $f(1)\leqq k\leqq f(N)-x$ and $f(N)-x<k\leqq f(N)$ and observe that if m and n are integers, $m\leqq n$, a and b are real numbers, $a<b$ such that $b>m$ and $n\geqq a$, then

$$\operatorname{card}([a,b)\cap[m,n]\cap\mathbf{Z}) = [\min(b,n)]-[\max(a,m)]+C_{\mathbf{Z}}(\max(a,m))-C_{(-\infty,n)\cap\mathbf{Z}}(b).$$

From this the proposition follows at once.

We get the following

COROLLARY 1. *Let $f\colon [1,\infty)\to\mathbf{R}$ be continuous and increasing in the strict sense. Let* $\lim\limits_{t\to\infty} f(t)=\infty$, $x\in[0,1]$ and $N\in\mathrm{N}$. Then

$$\sum_{n=1}^{N} C_{[0,x)}(\{f(n)\}) = \sum_{f(1)\leqq k\leqq f(N)} \big([f^{-1}(k+x)]-[f^{-1}(k)]+C_{\mathbf{Z}}(f^{-1}(k))-$$

$$-C_{\mathbf{Z}}(f^{-1}(k+x))\big)-C_{[0,x)}(\{f(N)\})\big(f^{-1}([f(N)]+x)-N\big)+O(1)\quad \textit{for}\quad N\to\infty.$$

The O-constant does not depend on x.

LEMMA 1. *Let $x\in[0,1]$, $\sigma\in(0,1)$, $\alpha>0$. Then we get*

$$\sum_{\alpha\leqq k\leqq\alpha N^{\sigma}}\left(\left(\frac{k+x}{\alpha}\right)^{1/\sigma}-\left(\frac{k}{\alpha}\right)^{1/\sigma}\right)-C_{[0,x)}(\{\alpha N^{\sigma}\})\big(\alpha^{-1/\sigma}([\alpha N^{\sigma}]+x)^{1/\sigma}-N\big)=$$

$$=xN+N^{1-\sigma}(\alpha\sigma)^{-1}\left(\frac{1}{2}(x+x^2)-x\{\alpha N^{\sigma}\}-(x-\{\alpha N^{\sigma}\})C_{[0,x)}(\{\alpha N^{\sigma}\})\right)+$$

$$+O(N^{1-2\sigma}+1)\quad \textit{for}\quad N\to\infty.$$

The O-constant does not depend on x.

PROOF. We have

$$(k+x)^{1/\sigma}-k^{1/\sigma} = \frac{x}{\sigma}k^{-1+1/\sigma}+\frac{x^2}{2\sigma}\left(\frac{1}{\sigma}-1\right)k^{-2+1/\sigma}+O(k^{-3+1/\sigma})$$

where the O-constant does not depend on x. Furthermore, if $\beta\neq-1$,

$$\sum_{k=1}^{T} k^{\beta} = \frac{T^{\beta+1}}{\beta+1}+\frac{T^{\beta}}{2}+O(1+T^{\beta-1})\quad \text{as}\quad T\to\infty.$$

Hence

$$\sum_{\alpha\leqq k\leqq\alpha N^{\sigma}} \alpha^{-1/\sigma}\big((k+x)^{1/\sigma}-k^{1/\sigma}\big) =$$

$$= xN+(\alpha\sigma)^{-1}N^{1-\sigma}\left(\frac{1}{2}(x+x^2)-x\{\alpha N^{\sigma}\}\right)+O(N^{1-2\sigma}+1).$$

Because of

$$\alpha^{-1/\sigma}([\alpha N^{\sigma}]+x)^{1/\sigma} = \big(N^{\sigma}+\alpha^{-1}(x-\{\alpha N^{\sigma}\})\big)^{1/\sigma} =$$

$$= N+N^{1-\sigma}(\alpha\sigma)^{-1}(x-\{\alpha N^{\sigma}\})+O(N^{1-2\sigma})$$

we get the result.

2. The discrepancy of the sequence (αn^{σ}) mod 1, $0<\sigma<\frac{1}{2}$, $\alpha>0$

PROPOSITION 2. *Let $0<\sigma<\frac{1}{2}$, $x\in[0,1]$, $\alpha>0$. Then the following estimation holds for $N\to\infty$:*

$$N^{\sigma}\left(\frac{1}{N}\sum_{n=1}^{N} C_{[0,x)}(\{\alpha n^{\sigma}\})-x\right) =$$

$$= (\alpha\sigma)^{-1}\left(\frac{1}{2}x(1+x)-x\{\alpha N^{\sigma}\}-(x-\{\alpha N^{\sigma}\})C_{[0,x)}(\{\alpha N^{\sigma}\})\right)+O(N^{2\sigma-1}).$$

The O-constant does not depend on x.

This follows immediately from Corollary 1 and Lemma 1.

COROLLARY 2. *Let $0<\sigma<\frac{1}{2}$, $\alpha>0$ and $(x_n)=(\alpha n^{\sigma})$* mod 1. *Then*

$$(1)\qquad 2\alpha\sigma N^{\sigma}D_N^{*}((x_n)) = \max\left(\{\alpha N^{\sigma}\}(1-\{\alpha N^{\sigma}\}),\ \frac{1}{4}-\{\alpha N^{\sigma}\}(1-\{\alpha N^{\sigma}\})\right)+$$

$$+O(N^{2\sigma-1})\quad \textit{as}\quad N\to\infty.$$

$$(2)\qquad 2\alpha\sigma N^{\sigma}D_N((x_n)) = \frac{1}{4}+O(N^{2\sigma-1})\quad \textit{as}\quad N\to\infty.$$

PROOF. We consider for any fixed z, $0\leqq z<1$ the function

$$f\colon[0,1]\to\mathbf{R},\quad f(x)=\frac{1}{2}x(1+x)-xz-(x-z)C_{[0,x)}(z).$$

We note that f is continuous,

$$\max_{x\in[0,1]} f(x)=\frac{1}{2}z(1-z) \quad\text{and}\quad \min_{x\in[0,1]} f(x)=\frac{1}{2}z(1-z)-\frac{1}{8}.$$

Therefore and because of

$$2\alpha\sigma N^\sigma D_N*((x_n)) = 2\max\left(\left|\max_{x\in[0,1]} f(x)\right|, \left|\min_{x\in[0,1]} f(x)\right|\right)+O(N^{2\sigma-1})$$

the result in (1) follows with $z=\{\sigma N^\sigma\}$ by Proposition 2.

Furthermore, by Proposition 2

$$\alpha\sigma N^\sigma\left(\frac{1}{N}\sum_{n=1}^{N} C_{[y,x)}(\{\alpha n^\sigma\})-(x-y)\right)=f(x)-f(y)$$

and therefore

$$2\alpha\sigma N^\sigma D_N((x_n)) = 2\max\left(\left|\max_{0\leqq y<x\leqq 1}(f(x)-f(y))\right|, \left|\min_{0\leqq y<x\leqq 1}(f(x)-f(y))\right|\right)+$$
$$+O(N^{2\sigma-1})=\frac{1}{4}+O(N^{2\sigma-1}).$$

COROLLARY 3. Let $0<\sigma<\frac{1}{2}$, $\alpha>0$, $(x_n)=(\alpha n^\sigma)$ mod 1. *Then*

$$\overline{\lim_{N\to\infty}}\, N^\sigma D_N^*((x_n))=\frac{1}{8\alpha\sigma},\quad \underline{\lim_{N\to\infty}}\, N^\sigma D_N^*((x_n))=\frac{1}{16\alpha\sigma},$$
$$\lim_{N\to\infty} N^\sigma D_N((x_n))=\frac{1}{8\alpha\sigma}.$$

This follows immediately from Corollary 2.

Hence the constants 1 and 2 in the general inequality $D_N^*((x_n))\leqq 1\cdot D_N((x_n))\leqq \leqq 2\cdot D_N^*((x_n))$ cannot be improved.

3. A note on the discrepancy of the sequence (αn^σ) mod 1, $\alpha>0$, $\frac{1}{2}<\sigma<1$

PROPOSITION 3. *Let* $\alpha>0$ *and* $\frac{1}{2}<\sigma<1$. *Then* $\lim_{N\to\infty} N^{1-\sigma} D_N((x_n))=0$.

PROOF. Let $\gamma\geqq 0$. Then the set $\{((\alpha^{-1}k+y)^{1/\sigma})_{k\in\mathbf{N}} | 0\leqq y\leqq\gamma\}$ is equi-uniformly distributed mod 1 (see [1]). Therefore

$$\lim_{N\to\infty} N^{-\sigma}\sum_{\alpha\leqq k\leqq\alpha N^\sigma}\left(\{\alpha^{-1/\sigma}k^{1/\sigma}\}-\{\alpha^{-1/\sigma}(k+x)^{1/\sigma}\}+\right.$$
$$\left.+C_{\mathbf{Z}}(\alpha^{-1/\sigma}k^{1/\sigma})-C_{\mathbf{Z}}(\alpha^{-1/\sigma}(k+x)^{1/\sigma})\right)=0$$

uniformly with respect to $x\in[0,1]$. This together with Corollary 1 and Lemma 1 leads to the statement made above. One could easily sharpen the estimate $D_N((x_n))= =O(N^{\sigma-1})$ to: for explicitly calculable $\varepsilon>0$: $D_N((x_n))=O(N^{\sigma-1-\varepsilon})$.

PROBLEM. Find the best possible $\varepsilon>0$.

4. The discrepancy of the sequence $(\alpha\sqrt{n})$ mod 1 for $\alpha>0$, $\alpha^2\notin\mathbf{Q}$

PROPOSITION 4. *Let* $\alpha>0$ *and* $x\in[0,1]$. *Then*

$$\alpha\sqrt{N}\left(\frac{1}{N}\sum_{n=1}^{N} C_{[0,x)}(\{\alpha\sqrt{n}\})-x\right)=x(1+x)-2x\{\alpha\sqrt{N}\}-$$
$$-2(x-\{\alpha\sqrt{N}\})C_{[0,x)}(\{\alpha\sqrt{N}\})+\frac{\alpha}{\sqrt{N}}\sum_{1\leqq k\leqq\alpha\sqrt{N}}\left(\{\alpha^{-2}k^2\}-\{\alpha^{-2}(k+x)^2\}+\right.$$
$$\left.+C_{\mathbf{Z}}(\alpha^{-2}k^2)-C_{\mathbf{Z}}(\alpha^{-2}(k+x)^2)\right)+O(N^{-1/2}),$$

where the O-constant does not depend on x.

This follows immediately from Corollary 1 and Lemma 1.

COROLLARY 4. *Let* $\alpha>0$ *and* $x\in[0, 1]$, $\alpha^2\notin\mathbf{Q}$. *Then the following estimate holds for* $N\to\infty$:

$$\alpha\sqrt{N}\left(\frac{1}{N}\sum_{n=1}^{N} C_{[0,x)}(\{\alpha\sqrt{n}\})-x\right)=x(1+x)-2x\{\alpha\sqrt{N}\}-$$
$$-2(x-\{\alpha\sqrt{N}\})C_{[0,x)}(\{\alpha\sqrt{N}\})+O(1)$$

uniformly with respect to x.

PROOF. This follows from the fact that the set $\{(\alpha^{-2}k^2+\beta k+\gamma)_{k\in\mathbf{N}} | \beta,\gamma\in\mathbf{R}\}$ is equi-uniformly distributed mod 1. Therefore

$$\lim_{N\to\infty} N^{-1/2}\sum_{1\leqq k\leqq\alpha\sqrt{N}}\left(\{\alpha^{-2}k^2\}-\{\alpha^{-2}(k+x)^2\}+C_{\mathbf{Z}}(\alpha^{-2}k^2)-C_{\mathbf{Z}}(\alpha^{-2}(k+x)^2)\right)=0$$

uniformly with respect to $x\in[0, 1]$. Now use Proposition 4.

COROLLARY 5. *Let* $\alpha>0$, $\alpha^2\notin\mathbf{Q}$, $(x_n)=(\alpha\sqrt{n})$ mod 1. *Then*

$$\text{(1)}\quad \alpha\sqrt{N}D_N^*((x_n))=\max\left(\{\alpha\sqrt{N}\}(1-\{\alpha\sqrt{N}\}), \frac{1}{4}-\{\alpha\sqrt{N}\}(1-\{\alpha\sqrt{N}\})\right)+O(1)$$

$$\text{(2)}\quad \alpha\sqrt{N}D_N((x_n))=\frac{1}{4}+O(1).$$

The proof is the same as in Corollary 2.

COROLLARY 6. *Let* $\alpha>0$, $\alpha^2\notin\mathbf{Q}$, $(x_n)=(\alpha\sqrt{n})$ mod 1. *Then*

$$\overline{\lim_{N\to\infty}}\sqrt{N}D_N^*((x_n))=\frac{1}{4\alpha},\quad \underline{\lim_{N\to\infty}}\sqrt{N}D_N^*((x_n))=\frac{1}{8\alpha}$$

and

$$\lim_{N\to\infty}\sqrt{N}D_N((x_n))=\frac{1}{4\alpha}.$$

5. A note on the case $(x_n)=(\alpha\sqrt{n})$ mod 1, where $\alpha>0$, $\alpha^2\in\mathbf{Q}$

This case seems to be more complicated. We only prove the following proposition we shall need later:

PROPOSITION 5. *Let* $\alpha>0$, $\alpha^2=\frac{p}{q}$, *where* g.c.d. $(p,q)=1$, $q\geqq 1$. *Let a be the greatest integer with* $a^2|p$. *Then we get, with* $x\in[0,1]$:

(1) *if* $x\notin\mathbf{Q}$ *or* $x=\frac{r}{s}$, g.c.d. $(r,s)=1$, $s\geqq 1$ *and* $s^2\nmid q$, *then*

$$\alpha\sqrt{N}\left(\frac{1}{N}\sum_{n=1}^{N}C_{[0,x)}(\{\alpha\sqrt{N}\})-x\right)=x(1+x)-2x\{\alpha\sqrt{N}\}-$$

$$-2(x-\{\alpha\sqrt{N}\})C_{[0,x)}(\{\alpha\sqrt{N}\})+\frac{a}{q}+\frac{1}{q}\sum_{k=1}^{p-1}\left\{\frac{k^2q}{p}\right\}-$$

$$-\frac{\alpha}{\sqrt{N}}\sum_{1\leqq k\leqq\alpha\sqrt{N}}\{\alpha^{-2}(k+x)^2\}+O(N^{-1/2}).$$

(2) *if* $x=\frac{r}{s}$, g.c.d. $(r,s)=1$, $s\geqq 1$, $s^2|q$, *then*

$$\alpha\sqrt{N}\left(\frac{1}{N}\sum_{n=1}^{N}C_{[0,x)}(\{\alpha\sqrt{N}\})-x\right)=x(1+x)-2x\{\alpha\sqrt{N}\}-$$

$$-2(x-\{\alpha\sqrt{N}\})C_{[0,x)}(\{\alpha\sqrt{N}\})+O(N^{-1/2}).$$

In both cases the O-constant does not depend on x.

PROOF. We have

$$\frac{\alpha}{\sqrt{N}}\sum_{1\leqq k\leqq\alpha\sqrt{N}}\{\alpha^{-2}k^2\}=\frac{1}{q}\sum_{k=1}^{p-1}\left\{\frac{k^2q}{p}\right\}+O(N^{-1/2})$$

and

$$\frac{\alpha}{\sqrt{N}}\sum_{1\leqq k\leqq\alpha\sqrt{N}}C_{\mathbf{Z}}(\alpha^{-2}k^2)=\frac{a}{q}+O(N^{-1/2}).$$

(1) If $x\notin\mathbf{Q}$ then there is at most one $k\in\mathbf{Z}$ with $\alpha^{-2}(k+x)^2\in\mathbf{Z}$. Therefore

$$\frac{\alpha}{\sqrt{N}}\sum_{1\leqq k\leqq\alpha\sqrt{N}}C_{\mathbf{Z}}(\alpha^{-2}(k+x)^2)=O(N^{-1/2})$$

uniformly with respect to x. If $x=\frac{r}{s}$ and $s^2\nmid q$, then there is no $k\in\mathbf{Z}$ with $\alpha^{-2}(k+x)^2\in\mathbf{Z}$. For otherwise $\frac{q}{p}\left(k+\frac{r}{s}\right)^2\in\mathbf{Z}$ and therefore $s|r^2q$ which would imply $s|q$. Furthermore $s^2|2\,skrq+r^2q$ and therefore $s^2|q$. Now the result follows from Proposition 4.

(2) Now let $s^2|q$. Then $p|(sk+r)^2$ if and only if $\left.\frac{p}{a}\right|sk+r$; hence $\alpha^{-2}(k+x)^2\in\mathbf{Z}$ if and only if $\left.\frac{p}{a}\right|sk+r$. This together with g.c.d. $\left(s,\frac{p}{a}\right)=1$ yields to the fact

$$\frac{\alpha}{\sqrt{N}}\sum_{1\leqq k\leqq\alpha\sqrt{N}}C_{\mathbf{Z}}(\alpha^{-2}(k+x)^2)=\frac{a}{q}+O(N^{-1/2})$$

uniformly with respect to x (in fact there is only a finite number of x with the required properties).

Furthermore

$$\frac{\alpha}{\sqrt{N}}\sum_{1\leqq k\leqq\alpha\sqrt{N}}\{\alpha^{-2}(k+x)^2\}=\frac{\alpha^2}{p}\sum_{k=1}^{p}\left\{\frac{(ks+r)^2}{p}\frac{q}{s^2}\right\}+O(N^{-1/2})=$$

$$=\frac{\alpha^2}{p}\sum_{k=0}^{p-1}\left\{\frac{k^2}{p}\frac{q}{s^2}\right\}+O\{N^{-1/2})=\frac{\alpha^2}{p}\sum_{k=0}^{p-1}\left\{\frac{k^2}{p}\right\}+O(N^{-1/2})$$

because of g.c.d. $(s,p)=$ g.c.d. $\left(p,\frac{q}{s^2}\right)=1$. The result follows from Proposition 4 again.

6. The discrepancy of the sequence $(\alpha\sqrt{n})$ mod 1 for $\alpha>0$, $\alpha^{-2}\in\mathbf{N}$

Let us introduce the abbreviation

$$\omega_N(x):=\alpha\sqrt{N}\left(\frac{1}{N}\sum_{n=1}^{N}C_{[0,x)}(\{\alpha\sqrt{N}\})-x\right).$$

We define for $q\in\mathbf{N}\,\psi_q$ by $\psi_q=\frac{1}{4}+\frac{1}{q}-\frac{3}{2q^2}$ if $q\equiv 0(4)$, $q\neq 4$, $\psi_4=\frac{53}{128}$. $\psi_q=\frac{1}{4}+\frac{7}{8q}$ if $q\equiv 1(4)$, $\psi_q=\frac{1}{4}+\frac{1}{q}-\frac{3}{8q^2}$ if $q\equiv 2(4)$ and $\psi_q=\frac{1}{4}+\frac{7}{8q}-\frac{3}{8q^2}$ if $q\equiv 3(4)$. We prove the following

PROPOSITION 6. *Let* $\alpha>0$, $\alpha^2=\frac{1}{q}$ *for some* $q\in\mathbf{N}$, $(x_n)=(\alpha\sqrt{n})$ mod 1. *Then*

$$\varlimsup_{N\to\infty}\alpha\sqrt{N}D_N^*((x_n))\geqq\psi_q.$$

PROOF. Let $\varepsilon_q=2$ for $q\equiv 0(4)$, $q\neq 4$; $\varepsilon_4=1$; $\varepsilon_q=0$ for $q\equiv 1(4)$; $\varepsilon_q=1$ for $q\equiv 2,3(4)$. Furthermore let

$$r_q=\left[\frac{q}{4}\left(1-\frac{\varepsilon_q}{q}\right)^2\right],\quad N_K=qK^2+(q-\varepsilon_q)K+r_q$$

for $K\in\mathbf{N}$ and $x_K=\{\alpha\sqrt{N_K}\}$. Then it is easily seen that $x_K\notin\mathbf{Q}$. We get

$$x_K=\frac{1}{2}\left(1-\frac{\varepsilon_q}{q}\right)+\frac{\varepsilon_q}{2qK}-\frac{1}{8}\left(1-\frac{\varepsilon_q}{q}\right)^2\frac{1}{K}+O\{K^{-2})\quad\text{as}\quad k\to\infty$$

by expanding the square-root into a power series. Furthermore for $1\leqq k<K$

$$\{2kqx_K+qx_K^2\}=1-\{2(K-k)qx_K\}.$$

Because of $K=[\alpha\sqrt{N_K}]$ and Proposition 5(1) we get:

$$\omega_{N_K}(x_K)=\frac{1}{4}\left(1-\frac{\varepsilon_q}{q^2}\right)+\frac{1}{qK}\sum_{k=1}^{K}\{2qkx_K\}+O(K^{-1}).$$

Now

$$\{2kqx_K\}=\left\{\frac{k}{K}\left(r_q-\frac{q}{4}\left(1-\frac{\varepsilon_q}{q}\right)^2\right)\right\}+O(K^{-1})$$

with at most $O(1)$ exceptions for k, $1\leqq k\leqq K$. This results in the fact that

$$\omega_{N_K}(x_K)=\frac{1}{4}\left(1-\frac{\varepsilon_q^2}{q^2}\right)+\frac{1}{qK}\sum_{k=1}^{K}\left\{-\frac{k}{K}\left\{\frac{q}{4}\left(1-\frac{\varepsilon_q}{q}\right)^2\right\}\right\}+O(K^{-1})=$$

$$=\frac{1}{4}\left(1-\frac{\varepsilon_q^2}{q^2}\right)+\frac{1}{q}-\frac{1}{2q}\left\{\frac{q}{4}\left(\frac{1}{2}-\frac{\varepsilon_q}{q}\right)\right\}+O(K^{-1}).$$

Now use $\alpha\sqrt{N_K}\,D^*_{N_K}((x_n))\geqq\omega_{N_K}(x_K)$ and

$$\psi_q=\frac{1}{4}\left(1-\frac{\varepsilon_q}{q^2}\right)+\frac{1}{q}-\frac{1}{2q}\left\{\frac{q}{4}\left(1-\frac{\varepsilon_q}{q}\right)^2\right\}.$$

LEMMA 2. *Let* $K\in\mathbf{N}$, $1\leqq m,\ b\leqq K$, $a\in\mathbf{N}$, g.c.d.$(a,b)=1$, $|\theta|<1$ *and* $\theta\notin\mathbf{Q}$. *Then*

$$\sum_{k=-m}^{K-m}\left\{\frac{ak}{b}+\frac{\theta k}{bK}\right\}=\frac{K}{2}-(1-|\theta|)\frac{K-2m}{2b}\operatorname{sgn}\theta-\frac{b}{2}\left(\left\{\frac{K-m}{b}\right\}-\left\{\frac{m}{b}\right\}\right)+$$

$$+\sum_{1\leqq k\leqq b\left\{\frac{K-m}{b}\right\}}\left\{\frac{ak}{b}\right\}-\sum_{1\leqq k\leqq b\left\{\frac{m}{b}\right\}}\left\{\frac{ak}{b}\right\}+O(1),$$

where the O-constant is absolute.

The proof is a straightforward application of well known ideas so that we can omit it.

LEMMA 3. *Let* $K\in\mathbf{N}$, $1\leqq m,\ b\leqq K$, $a\in\mathbf{N}$, g.c.d. $(a,b)=1$, $|\theta|<1$ *and* $\theta\notin\mathbf{Q}$. *Then*

$$\sum_{k=-m}^{K-m}\left\{\frac{ak}{b}+\frac{\theta k}{bK}\right\}\geqq\frac{K}{2}-(1-|\theta|)\frac{K-2m}{2b}\operatorname{sgn}\theta-\frac{b}{8}+O(1).$$

PROOF. Let $A=\left\{\frac{K-m}{b}\right\}$, $B=\left\{\frac{m}{b}\right\}$. Then we get:

$$-\frac{b}{2}\left(\left\{\frac{K-m}{b}\right\}-\left\{\frac{m}{b}\right\}\right)+\sum_{1\leqq k\leqq b\left\{\frac{K-m}{b}\right\}}\left\{\frac{ak}{b}\right\}-\sum_{1\leqq k\leqq b\left\{\frac{m}{b}\right\}}\left\{\frac{ak}{b}\right\}=$$

$$=-\frac{b}{2}(A-B)+\operatorname{sgn}(A-B)\sum_{k=b\min(A,B)+1}^{b\max(A,B)}\left\{\frac{ak}{b}\right\}\geqq$$

$$\geqq-\frac{b}{2}(A-B)+\operatorname{sgn}(A-B)\sum_{k=1}^{b|A-B|}\left(\frac{1}{2}(1-\operatorname{sgn}(A-B))+\operatorname{sgn}(A-B)\frac{k}{b}\right)=$$

$$=-\frac{b}{2}(A-B)+\frac{1}{2}b(\operatorname{sgn}(A-B)-1)|A-B|+\frac{b}{2}(A-B)^2+O(1)=$$

$$=\frac{b}{2}\left((A-B)^2-|A-B|\right)+O(1)\geqq-\frac{b}{8}+O(1)$$

because of $|A-B|\leqq 1$. Now use Lemma 2.

LEMMA 4. *Let* $N, q, n\in\mathbf{N}$, $1\leqq n\leqq N$, $m=\left[\sqrt{\frac{n}{q}}\right]$, $K=\left[\sqrt{\frac{N}{q}}\right]$, $a, b\in\mathbf{N}$, g.c.d. $(a,b)=1$, $1\leqq b\leqq K$, $\theta\notin\mathbf{Q}$ *and* $|\theta|<1$.

(1) *If* $4b^2q\nmid a^2$ *or* $\theta>0$, *then* $\frac{|\theta|}{K}>\frac{b}{2(m+1)}\left|1+\operatorname{sgn}\theta-2\left\{\frac{a^2}{4b^2q}\right\}\right|$.
(2) *If* $4b^2q\mid a^2$ *and* $\theta<0$, *then* $\theta=-1+O(K^{-1})$.

PROOF. First we note that there is at most one n which lies between $\frac{a}{b}$ and $\frac{a}{b}+\frac{\theta}{bK}$. Therefore, if $\theta>0$ $(\theta<0)$, $4b^2nq$ is the smallest (largest) integer which is greater (smaller) than a^2 and which is divisible by $4b^2q$.

Hence, if $4b^2q\nmid a^2$, $4b^2nq=4b^2qt$, where

$$t=\left[\frac{a^2}{4b^2q}\right]+\frac{1}{2}(1+\operatorname{sgn}\theta).$$

It follows that

$$4b^2nq-a^2=2b^2q(1+\operatorname{sgn}\theta)-4b^2q\left\{\frac{a^2}{4b^2q}\right\}$$

and

$$\frac{|\theta|}{K}=|2b\sqrt{nq}-a|=(2b\sqrt{2}q+a)^{-1}\left|2b^2q(1+\operatorname{sgn}\theta)-4b^2q\left\{\frac{a^2}{4b^2q}\right\}\right|.$$

Now $(m+1)q>\sqrt{nq}$ and because of

$$\sqrt{\frac{n}{q}}=\frac{a}{2bq}+\frac{\theta}{2bqK}\quad\text{and}\quad\frac{a}{2bq}\notin\mathbf{Z}$$

or $\theta>0$, $m=\left[\frac{a}{2bq}\right]$. Therefore

$$(m+1)q>\frac{a}{2b}\quad\text{and}\quad 2b\sqrt{nq}+a<4bq(m+1).$$

Now let us assume that $4b^2q|a^2$ and $\theta<0$. Then $4b^2nq=4b^2qt$, where $t=\frac{a^2}{4b^2q}-1$ and $4b^2nq-a^2=-4b^2q$. Therefore

$$\left|2b\sqrt{nq}-a\right|=(2b\sqrt{nq}+a)^{-1}4b^2q.$$

Now $\frac{a}{2bq}<m+2$ and $\sqrt{nq}<(m+1)q$, such that $\frac{|\theta|}{K}>(m+2)^{-1}$. Hence $m=K-1$ or $m=K$ and the result follows.

Lemma 5. *Let* $N, q, n, a\in\mathbf{N}$, $1\leqq n\leqq N$, $m=\left[\sqrt{\frac{n}{q}}\right]$, $K=\left[\sqrt{\frac{N}{q}}\right]$, $2\sqrt{nq}=a+\frac{\theta}{K}$ *where* $\theta\notin\mathbf{Q}$, $|\theta|<1$. *Then*

$$\left\{\frac{a}{2q}\right\}\left(1-\left\{\frac{a}{2q}\right\}\right)+\frac{1}{q}-\frac{1}{qK}\sum_{k=-m}^{K-m}\left\{\frac{\theta k}{K}\right\}\leqq\psi_q+O(K^{-1}).$$

Proof. Because of $\frac{1}{q}\leqq\psi_q$ there is nothing to prove if $2q|a$. Therefore we may assume that $2q\nmid a$. Then $\left[\frac{a}{2q}\right]=m$. Hence there exists an ε, $|\varepsilon|\leqq q$ such that $a=2qm+q+\varepsilon$.

We set $b=1$ in Lemma 3. Then we get:

$$\left\{\frac{a}{2q}\right\}\left(1-\left\{\frac{a}{2q}\right\}\right)+\frac{1}{q}-\frac{1}{qK}\sum_{k=-m}^{K-m}\left\{\frac{\theta k}{K}\right\}\leqq$$

$$\leqq\frac{1}{4}-\frac{\varepsilon^2}{4q^2}+\frac{1}{q}-\frac{1}{2q}-(1-|\theta|)\frac{2m-K}{2}\operatorname{sgn}\theta=$$

$$=\frac{1}{4}+\frac{1}{2q}-\frac{\varepsilon^2}{4q^2}+(1-|\theta|)\frac{-2m+K}{2qK}\operatorname{sgn}\theta=:A(\theta,K,q,m,\varepsilon).$$

If $\operatorname{sgn}\theta\neq\operatorname{sgn}(K-2m)$, we get

$$A(\theta,K,q,m,\varepsilon)\leqq\frac{1}{4}+\frac{1}{2q}\leqq\psi_q$$

so that we are done.

Therefore we may assume that $\operatorname{sgn}\theta=\operatorname{sgn}(K-2m)$. Then we get

$$A(\theta,K,q,m,\varepsilon)\leqq\frac{1}{4}+\frac{1}{q}-\frac{\varepsilon^2}{4q^2}-\frac{|\theta|}{2q}\leqq\frac{1}{4}+\frac{1}{q}-\frac{\varepsilon^2}{4q^2}-$$

$$-\frac{K}{4q(m+1)}\left|1+\operatorname{sgn}\theta-2\left\{\frac{(q+\varepsilon)}{4q}\right\}\right|\quad\text{if}\quad 4q\nmid a^2\quad\text{or}\quad\theta>0$$

by Lemma 4. If $4q|a^2$, then

$$A(\theta,K,q,m,\varepsilon)\leqq\frac{1}{4}+\frac{1}{2q}<\psi_q$$

so that we are done. Therefore we may assume that $4q\nmid a^2$. Now

$$\frac{K}{m+1}\geqq 1+\frac{1}{2}(1+\operatorname{sgn}\theta)-O(K^{-1})$$

and therefore

$$A(\theta,K,q,m,\varepsilon)\leqq\frac{1}{4}+\frac{1}{q}-\frac{\varepsilon^2}{4q^2}-\frac{1}{2q}\left(1+\frac{1}{2}(1+\operatorname{sgn}\theta)\right)\times$$

$$\times\left|(1+\operatorname{sgn}\theta)\frac{1}{2}-\left\{\frac{q}{4}+\frac{\varepsilon}{2}+\frac{\varepsilon^2}{4q}\right\}\right|=:B(q,\theta,\varepsilon).$$

(1) Let $q\equiv 0\,(4)$. Then $\varepsilon\neq 0$.

(1a) Let $q\neq 4$. Then

$$B(q,\theta,\varepsilon)\leqq\frac{1}{4}+\frac{1}{q}-\frac{\varepsilon^2}{4q^2}-\frac{1}{2q}\left|(1+\operatorname{sgn}\theta)\frac{1}{2}-\left\{\frac{\varepsilon}{2}+\frac{\varepsilon^2}{4q}\right\}\right|$$

so that we are done in the case $|\varepsilon|\geqq 3$. If $|\varepsilon|=1$ we get

$$B(q,\theta,\varepsilon)\leqq\frac{1}{4}+\frac{1}{q}-\frac{1}{4q^2}-\frac{1}{2q}\left(\frac{1}{2}-\frac{1}{4q}\right)\leqq\psi_q.$$

If $|\varepsilon|=2$ we get

$$B(q,\theta,\varepsilon)\leqq\frac{1}{4}+\frac{1}{q}-\frac{3}{2q^2}=\psi_q.$$

(1b) Let $q=4$. Then

$$B(4,\theta,\varepsilon)\leqq\frac{1}{2}-\frac{\varepsilon^2}{64}\leqq\psi_4\quad\text{if}\quad|\varepsilon|\geqq 3.$$

Therefore we may assume $|\varepsilon|\leqq 2$. If $|\varepsilon|=1$,

$$B(4,\theta,\varepsilon)=\frac{1}{2}-\frac{1}{64}-\frac{1}{8}\left(1+\frac{1}{2}(1+\operatorname{sgn}\theta)\right)\left|(1+\operatorname{sgn}\theta)\frac{1}{2}-\left\{\frac{1}{2}+\frac{1}{16}\right\}\right|\leqq\psi_4.$$

If $|\varepsilon|=2$ we get

$$B(4,\theta,\varepsilon)=\frac{1}{2}-\frac{1}{16}-\frac{1}{8}\left(1+\frac{1}{2}(1+\operatorname{sgn}\theta)\right)\left|(1+\operatorname{sgn}\theta)\frac{1}{2}-\frac{1}{4}\right|\leqq\frac{1}{2}-\frac{1}{16}-\frac{1}{32}<\psi_4.$$

(2) Let $q\equiv 1\,(4)$. Then

$$B(q,\theta,\varepsilon)\leqq\frac{1}{4}+\frac{1}{q}-\frac{\varepsilon^2}{4q^2}-\frac{1}{2q}\left|(1+\operatorname{sgn}\theta)\frac{1}{2}-\left\{\frac{1}{4}+\frac{\varepsilon}{2}+\frac{\varepsilon^2}{4q}\right\}\right|\leqq\psi_q\quad\text{if}\quad\varepsilon^2\geqq\frac{q}{2}.$$

Therefore we may assume that $\varepsilon^2<\frac{q}{2}$. If $\varepsilon\equiv 0\,(2)$ we get

$$B(q,\theta,\varepsilon)\leqq\frac{1}{4}+\frac{1}{q}-\frac{\varepsilon^2}{4q^2}-\frac{1}{2q}\left(\frac{1}{4}+\frac{\varepsilon^2}{4q}\right)\leqq\psi_q.$$

If $\varepsilon\equiv 1(2)$ we get

$$B(q,\theta,\varepsilon)\leqq\frac{1}{4}+\frac{1}{q}-\frac{\varepsilon^2}{4q^2}-\frac{1}{2q}\left(\frac{1}{4}-\frac{\varepsilon^2}{4q}\right)\leqq\psi_q.$$

(3) Let $q\equiv 2(4)$. Then

$$B(q,\theta,\varepsilon)\leqq\frac{1}{4}+\frac{1}{q}-\frac{\varepsilon^2}{4q^2}-\frac{1}{2q}\left|(1+\operatorname{sgn}\theta)\frac{1}{2}-\left\{\frac{1}{2}+\frac{\varepsilon}{2}+\frac{\varepsilon^2}{4q}\right\}\right|\leqq\psi_q \quad\text{if}\quad \varepsilon^2\geqq 2.$$

Therefore we may assume that $|\varepsilon|\leqq 1$. If $\varepsilon=0$, then

$$B(q,\theta,0)\leqq\frac{1}{4}+\frac{1}{q}-\frac{1}{4q}<\psi_q.$$

If $|\varepsilon|=1$, then

$$B(q,\theta,\varepsilon)\leqq\frac{1}{4}+\frac{1}{q}-\frac{3}{8q^2}=\psi_q.$$

(4) Let $q\equiv 3(4)$. Then

$$B(q,\theta,\varepsilon)=\frac{1}{4}+\frac{1}{q}-\frac{\varepsilon^2}{4q^2}-\frac{1}{2q}\left(1+\frac{1}{2}(1+\operatorname{sgn}\theta)\right)\left|(1+\operatorname{sgn}\theta)\frac{1}{2}-\left\{\frac{3}{4}+\frac{\varepsilon}{2}+\frac{\varepsilon^2}{4q}\right\}\right|$$

so that we are done in the case $\varepsilon^2\geqq\frac{q+3}{2}$. Therefore we may assume that $\varepsilon^2\leqq\frac{q+1}{2}$. If $\varepsilon\equiv 0(2)$ and $\theta>0$ we get

$$B(q,\theta,\varepsilon)\leqq\frac{1}{4}+\frac{1}{q}-\frac{\varepsilon^2}{4q^2}-\frac{1}{q}\left(\frac{1}{4}-\frac{\varepsilon^2}{4q}\right)\leqq\psi_q.$$

If $\theta<0$ we get

$$B(q,\theta,\varepsilon)\leqq\frac{1}{4}+\frac{1}{q}-\frac{\varepsilon^2}{4q^2}-\frac{1}{2q}\left(\frac{3}{4}+\frac{\varepsilon^2}{4q}\right)\leqq\psi_q.$$

If $\varepsilon\equiv 1(2)$ then

$$B(q,\theta,\varepsilon)\leqq\frac{1}{4}+\frac{1}{q}-\frac{\varepsilon^2}{4q^2}-\frac{1}{2q}\left(\frac{1}{4}+\frac{\varepsilon^2}{4q}\right)\leqq\psi_q.$$

LEMMA 6. *Let $q,n,N\in\mathbf{N}$, $n\leqq N$ such that $x=\left\{\sqrt{\frac{n}{q}}\right\}\notin\mathbf{Q}$. Let $K=\left[\sqrt{\frac{N}{q}}\right]$. Then*

$$x(1-x)+\frac{1}{q}-\frac{1}{qK}\sum_{k=1}^{K}\{2kqx+qx^2\}\leqq\psi_q+O(K^{-1})$$

where the O-constant does not depend on n and N.

PROOF. Let $m=\left[\sqrt{\frac{n}{q}}\right]$. Then

$$\sum_{k=1}^{K}\{2kqx+qx^2\}=\sum_{k=1-m}^{K-m}\{2k\sqrt{nq}\}.$$

Now we use Dirichlet's theorem: there are natural numbers a and b with g.c.d. $(a,b)=1$, $1\leqq b\leqq K$ such that $2\sqrt{n}\,q=\frac{a}{b}+\frac{\theta}{bK}$ for some θ, $|\theta|<1$, $\theta\notin\mathbf{Q}$. Because of Lemma 3 we get

$$\sum_{k=1}^{K}\{2qkx+qx^2\}\geqq\frac{K}{2}-\frac{b}{8}-(1-|\theta|)\frac{K-2m}{2b}\operatorname{sgn}\theta.$$

First we assume that $b\geqq 2$. Then

$$x(1-x)+\frac{1}{q}-\frac{1}{qK}\sum_{k=1}^{K}\{2kqx+qx^2\}\leqq\frac{1}{4}+\frac{1}{q}-\frac{1}{2q}+\frac{b}{8qK}+$$

$$+(1-|\theta|)\frac{|K-2m|}{2bqK}\leqq\frac{1}{4}+\frac{1}{2q}+\frac{b}{8qK}+\frac{1}{2bq}\leqq\psi_q\quad\text{if}\quad q\neq 4\quad\text{or if}\quad b\geqq 4.$$

Hence we may assume that $q=4$ and $b=2$ (resp. $=3$). Then $4b^2q\nmid a^2$ and $m=\left[\frac{a}{2bq}\right]$. There is an $\varepsilon\in\mathbf{Z}$ such that $a=2bqm+bq+\varepsilon$ and $|\varepsilon|\leqq bq$. Furthermore

$$x=\left\{\frac{a}{2bq}\right\}+O(K^{-1})=\frac{1}{2}+\frac{\varepsilon}{2bq}+O(K^{-1}).$$

We have to prove that

$$\frac{3}{8}-\frac{\varepsilon^2}{4b^2q^2}+\frac{1}{8b}-\frac{|\theta|}{8b}\leqq\frac{53}{128}.$$

This is trivial if $\varepsilon^2\geqq 8b-\frac{5}{2}b^2$ so that we may assume that $\varepsilon^2<8b-\frac{5}{2}b^2$, that is $|\varepsilon|\leqq 2$ (resp. $|\varepsilon|\leqq 1$). Now

$$\frac{|\theta|}{b}\geqq\left|(1+\operatorname{sgn}\theta)\frac{1}{2}-\left\{\frac{m\varepsilon}{b}+\frac{\varepsilon}{2b}+\frac{\varepsilon^2}{16b^2}\right\}\right|$$

and $\varepsilon\neq 0$. If $b=2$ we get $\frac{|\theta|}{b}\geqq\frac{15}{64}$ and therefore

$$\frac{3}{8}-\frac{\varepsilon^2}{4b^2q^2}+\frac{1}{8b}-\frac{|\theta|}{8b}\leqq\frac{3}{8}-\frac{1}{4\cdot 64}+\frac{1}{16}-\frac{15}{8\cdot 64}<\psi_4.$$

If $b=3$ we get $\frac{|\theta|}{b}\geqq\frac{23}{144}$ and therefore

$$\frac{3}{8}-\frac{\varepsilon^2}{4b^2q^2}+\frac{1}{8b}-\frac{|\theta|}{8b}\leqq\frac{3}{8}-\frac{1}{64\cdot 9}+\frac{1}{24}-\frac{23}{8\cdot 144}<\psi_q.$$

If $b=1$ we get $x=\left\{\frac{\varepsilon}{2q}\right\}+O(K^{-1})$. Furthermore

$$\sum_{k=1}^{K}\{2kqx+qx^2\}=\sum_{k=1-m}^{K-m}\left\{\frac{\theta k}{K}\right\}$$

so that we can use Lemma 5 to get the result.

THEOREM 1. *Let* $q\in \mathbf{N}$, $\omega_q=\left(\sqrt{\frac{n}{q}}\right)_{n\in \mathbf{N}}$ mod 1. Then

$$\varlimsup_{N\to\infty} \sqrt{N} D_N^*(\omega_q) = \sqrt{q}\left(\frac{1}{4}+\frac{1}{q}-\frac{3}{2q^2}\right)$$

if $q\equiv 0(4), q\neq 4$; $=\frac{53}{128}$ *if* $q=4$; $=\sqrt{q}\left(\frac{1}{4}+\frac{7}{8q}\right)$ *if* $q\equiv 1(4)$; $=\sqrt{q}\left(\frac{1}{4}+\frac{1}{q}-\frac{3}{8q^2}\right)$ *if* $q\equiv 2(4)$; $\sqrt{q}\left(\frac{1}{4}+\frac{7}{8q}-\frac{3}{8q^2}\right)$ *if* $q\equiv 3(4)$.

PROOF. It is enough to prove that $|\omega_N(x)|\leqq\psi_q$ if $x\in[0,1]$ by Proposition 6. First we prove $\omega_N(x)\leqq\psi_q$. Let $\alpha=\frac{1}{\sqrt{q}}$, $K=\left[\sqrt{\frac{N}{q}}\right]$. If $x\notin \mathbf{Q}$ we get by Proposition 5:

$$\omega_N(x) = x(1+x)-2x\{\alpha\sqrt{N}\}-2(x-\{\alpha\sqrt{N}\})\, C_{[0,x)}(\{\alpha\sqrt{N}\})+$$

$$+\frac{1}{q}-\frac{1}{qK}\sum_{k=1}^{K}\{2kqx+qx^2\}+O(K^{-1})\leqq$$

$$\leqq x(1-x)+\frac{1}{q}-\frac{1}{qK}\sum_{k=1}^{K}\{2qkx+x^2q\}+O(K^{-1}).$$

ω_N is continuous from the left. Let $\{y_1, \dots, y_t\}=\left\{\{\alpha\sqrt{n}\}\,\middle|\,1\leqq n\leqq N\right\}$, where $y_i<y_{i+1}$ for $1\leqq i\leqq t$, $y_{t+1}=1$. Then $y_1=0$ (if $N\geqq q$). Because of the fact that $\omega_N|(y_i, y_{i+1}]$ ist monotone decreasing we get:

$$\sup_{x\in[0,1]} \omega_N(x)\in\left\{\lim_{x\to\{\alpha\sqrt{n}\}+0} \omega_N(x)\,|\,1\leqq n\leqq N\right\}.$$

Let for $1\leqq i\leqq t$ a_i be the number of n's, $1\leqq n\leqq N$, such that $\{\alpha\sqrt{n}\}=y_i$. Then $a_i=1$ for $1<i\leqq t$ and $a_1=K$. Furthermore

$$\lim_{x\to y_i+0}\omega_N(x) = \alpha\sqrt{N}\left(\frac{1}{N}\sum_{j=1}^{i}a_j-y_i\right)$$

and

$$\omega_N(y_i) = \alpha\sqrt{N}\left(\frac{1}{N}\sum_{j=1}^{i-1}a_j-y_i\right).$$

Therefore

$$\lim_{x\to y_i+0}\omega_N(x)-\omega_N(y_i) = \alpha\frac{a_i}{\sqrt{N}}.$$

By Proposition 6

$$\sup_{x\in[0,1]}\omega_N(x) > \lim_{x\to 0+0}\omega_N(x) = \frac{1}{q}+O\left(\frac{1}{K}\right).$$

Therefore

$$\sup_{0\leqq x\leqq 1}\omega_N(x)\in\left\{\lim_{x\to y_i+0}\omega_N(x)|2\leqq i\leqq t\right\}=$$

$$=\left\{\omega_N(\{\alpha\sqrt{n}\})+O(K^{-1})|1\leqq n\leqq N, \{\alpha\sqrt{n}\}\notin\mathbf{Q}\right\}.$$

Now we can use Lemma 6.

Let us prove $\omega_N(x)\geqq-\psi_q$. By Proposition 5 and what has been said above

$$\inf_{0\leqq x\leqq 1}\omega_N(x)\geqq x(1+x)-2x\{\alpha\sqrt{N}\}-$$

$$-2(x-\{\alpha\sqrt{N}\})\, C_{[0,x)}(\{\alpha\sqrt{N}\})\geqq-\frac{1}{4}\geqq-\psi_q.$$

This yields the theorem above.

PROPOSITION 7. *Let* $q\in\mathbf{N}$, $1\leqq q\leqq 4$. *Then*

$$\varliminf_{N\to\infty}\sqrt{N}D_N^*(\omega_q)=\frac{1}{\sqrt{q}},\quad \textit{where}\quad \omega_q=\left(\sqrt{\frac{n}{q}}\right)\ \text{mod } 1.$$

PROOF. At the end of the proof of Theorem 1 we showed that $\omega_N(x)\geqq-\frac{1}{4}\geqq-\frac{1}{q}$. Furthermore, if $N_K=qK^2$ for $K\in\mathbf{N}$, we see from Proposition 6 that

$$\omega_{N_K}(x) = x(1+x)-2x+\frac{1}{q}-\frac{1}{qK}\sum_{k=1}^{K}\{2kqx+qx^2\}\leqq\frac{1}{q}.$$

Therefore $|\omega_{N_K}(x)|\leqq\frac{1}{q}$ and $\varliminf_{N\to\infty}\alpha\sqrt{N}D_N^*(\omega_q)\leqq\frac{1}{q}$. Furthermore for each N, $\lim_{x\to 0+0}\omega_N(x)\sim\frac{1}{q}$, so that $\varliminf_{N\to\infty}\alpha\sqrt{N}D_N^*(\omega_q)\geqq\frac{1}{q}$.

CONJECTURE. Proposition 7 is valid at least in the case $q=5$ too (and perhaps in some further cases $q=6, \dots$). But if $q=10$, we conjecture that $\varliminf_{N\to\infty}\sqrt{N}D_N^*(\omega_q)=\frac{3}{2\sqrt{10}}$.

REMARK. The guess of the result of Theorem 1 was essentially facilitated by numerical computations. In this connection the author is particularly grateful to E. Dittrich and W. Pechlaner, who carried out these calculations.

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ON FINITELY PROJECTED MODULAR LATTICES

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1. Introduction

Let **K** be a variety of lattices. A lattice L in **K** is called finitely **K**-projected (see A. Day [1]) if for any surjection $f\colon A \twoheadrightarrow L$ in **K** there is a finite sublattice of A whose image under f is L; i.e. if L is the epimorphic image of a finite projective lattice in **K**. Let **M** be the variety of all modular lattices. In this paper we give a necessary condition for a lattice L to be finitely **M**-projected.

2. Results

In order to formulate our theorems we need some definitions. We call an ordered five-tuple (v, x, y, z, u) of elements from a modular lattice a *diamond* if these elements form a copy of M_3 with v and u as the bottom and the top elements, respectively. If a/b and c/d are quotients in a lattice we write $a/b \nearrow c/d$ and we say that a/b *transposes* up to c/d if $a \wedge d = b$ and $a \vee d = c$. In this case we also say that c/d transposes down to a/b, written $c/d \searrow a/b$. We also say that a/b and c/d are transposed intervals. The quotients a/b, c/d are said to be *projective* — in symbol $a/b \approx c/d$ — if there exists a sequence of quotients $a/b = a_0/b_0, a_1/b_1, \dots, a_n/b_n = c/d$ such that a_k/b_k and a_{k+1}/b_{k+1} are transposed intervals for every $k = 0, 1, \dots, n-1$. A sublattice K of L is called an *isometric* sublattice if a prime quotient in K is a prime quotient in L.

Let (v, x, y, z, u) be an isometric diamond of $L \in \mathbf{M}$. Let y be a double-irreducible element. Then $L' = L \setminus \{y\}$ is a sublattice. In [3] we have proved that if x/v and z/v are projective in L' then L is not splitting modular, i.e. not finitely **M**-projected.

Theorem 1. *Let (v, x, y, z, u) be an isometric diamond of a finitely* **M**-*projected lattice L such that y is meet-irreducible and $t \vee v = u$ implies $t = u$. Then $L' = L \setminus y^*$ is a sublattice of L, where $y^* = \{t;\, t \vee v = y\}$, and the quotients x/v and z/v are not projective in the sublattice L'.*

The diamond $D_1 = (v_1, x_1, y_1, z_1, u_1)$ is said to be *translate up* to the diamond $D_2 = (v_2, x_2, y_2, z_2, u_2)$ if one of the quotients u_1/x_1, u_1/y_1, u_1/z_1 transposes up to one of the quotients x_2/v_2, y_2/v_2, z_2/v_2 and we write $D_1 \nearrow D_2$. In this situation we also say that D_2 translate down to D_1, written $D_2 \searrow D_1$. Then $D_1 \sim D_2$ means that either $D_1 \nearrow D_2$ or $D_1 \searrow D_2$.

A sequence $D_0, D_1, \dots, D_{n-2}$ is called a *diamond circle* ([2]) if the following are satisfied:

(i) $n \geqq 4$

(ii) $D_i \sim D_{i+2}$ and $D_i \nsim D_{i+2}$ hold for every $i \in Z_n$.

Let $\mathbf{B}_2$ be the variety generated by all modular lattices of breadth two.

THEOREM 2. *Let L be a finite modular lattice of breadth two. If L is finitely $\mathbf{B}_2$-projected then L does not contain a diamond circle or a sublattice isomorphic to M_4.*

As a corollary of Theorem 2 from a result of A. MITSCHKE and R. WILLE [2] we get that in a finitely $\mathbf{B}_2$-projected lattice every prime quotient is stable.

3. Function lattices

Let L be a lattice and P be a partially ordered set. L^P denotes the lattice of all orderpreserving maps of P to L partially ordered by $f \leqq g$ if and only if $f(x) \leqq g(x)$ for each $x \in P$. L^P is called a function lattice. If $a \in L$ then $\bar{a}$ denotes the corresponding constant mapping, i.e. $\bar{a}(x)=a$ for each $x \in P$. The constant mappings form a sublattice of L^P which is isomorphic to L; we identify L with this sublattice. If a/b is a prime quotient of L then the corresponding quotient $a//b=\bar{a}/\bar{b}$ of L^P is isomorphic to $\underset{\sim}{2}^P$, where $\underset{\sim}{2}$ denotes the two element lattice. We have therefore a natural isomorphism $\varepsilon_{ab}: 2^P \to a//b$. Take all quotients $a//b$ of L^P where a/b runs over all prime quotients of L. All these generate a sublattice $L[2^P]$ of L^P. This $L[\underset{\sim}{2}^P]$ is a subdirect power of L. Take an arbitrary congruence relation θ of $\underset{\sim}{2}^P$ then θ can be extended to $L[2^P]$.

If L is finite then $L[2^P]$ is locally finite. In this paper P denotes the chain $\omega \cong \{0, 1, 2, \dots\}$. Then $\underset{\sim}{2}^\omega \cong 1+\omega^*$.

4. Proof of Theorem 1

Let a/b and c/d be projective puotients of a finite lattice L. We can assume that the corresponding sequence of transposed intervals alternate up and down, i.e.

$$a/b = a_0/b_0 \nearrow a_1/b_1 \searrow a_2/b_2 \nearrow \dots a_n/b_n = c/d \tag{1}$$

or conversely.

Let M be a lattice and assume there is a surjection $f: M \twoheadrightarrow L$. If there are elements $a_0', b_0', a_1', b_1', \dots, a_n', b_n'$ in M such that $f(a_i')=a_i, f(b_i')=b_i$ and a_i'/b_i', a_{i+1}'/b_{i+1}' are transposed for every i then we say that (1) has an inverse in M. The inverse of (1) is the sequence $a_0'/b_0' \nearrow a_1'/b_1' \searrow \dots a_n'/b_n'$. Let L be finitely projected in a variety $\mathbf{K} \subseteqq \mathbf{M}$. Then $M \in \mathbf{K}$ contains a finite sublattice M' whose image under f is L. We define a sequence of elements in M' which gives the inverse of (1). Let a_i' $(i=0, 2, 4, \dots)$ be the least element of M' whose image under f is a_i. Similarly b_i' $(i=1, 3, \dots)$ is the greatest element of M' such that $f(b_i')=b_i$. Further we set $b_0'=a_0' \wedge b_1'$, $a_1'=a_0' \vee b_1'$, $b_2'=b_1' \wedge a_2'$, $a_3'=a_2' \vee b_3'$, … . Then it is easy to see that

$$a_0'/b_0' \nearrow a_1'/b_1' \searrow a_2'/b_2' \nearrow \dots a_n'/b_n'.$$

We have therefore

LEMMA 1. *Let $f: M \twoheadrightarrow L$ be a surjection onto a finite lattice L in a variety $\mathbf{K} \subseteqq \mathbf{M}$. If there is a projectivity in L which has no inverse in M then L is not finitely $\mathbf{K}$-projected.*

Let (v, x, y, z, u) be an isometric diamond of the finitely $\mathbf{M}$-projected (or $\mathbf{B}_2$-projected) modular lattice such that y is meet-irreducible and $t \vee v=u$ implies $t=u$. Then we prove that $L'=L \setminus y^*$ is a sublattice, where $y^*=\{a;\ a \vee v=y\}$. Let $b \vee c \in y^*$, i.e. $b \vee c \vee v=y$. Then $v \leqq b \vee v \leqq y$ (resp. $v \leqq c \vee v \leqq y$) implies that $b \vee v$ (resp. $c \vee v$) is either v or y (y/v is a prime quotient). $b \vee v=c \vee v=v$ would imply $b \vee c \vee v=v$ hence we have $b \vee v=y$ or $c \vee v=y$, i.e. one of b and c is in y^*. On the other hand we assume that $d \wedge e \in y^*$, $d, e \notin y^*$ and $d \wedge e$ is covered by d and e. Then y is covered by $d \vee y$. If $d \vee y \neq u$ then $(d \vee y) \wedge u=y$, which is a contradiction to the meet-irreducibility of y. We have therefore $y \vee d=y \vee e=u$. Then $v \vee d=v \vee (d \wedge e) \vee d=y \vee d=u$ which yields by our assumption that $d=u$. Similarly $e=u$, i.e. $d \wedge e=u$ which is a contradiction to $d \wedge e \in y^*$.

First we take the extension $A=L'[2^\omega]$ of L'. Then x/v in A is isomorphic to $1+\omega^*$, i.e. x/v is the following chain

$$x = x_0 \succ x_1 \succ x_2 \succ \dots > v.$$

Similarly z/v is the chain $z=z_0 \succ z_1 \succ z_2 \succ \dots > v$. Set $w_0=x_1 \vee z_0$, $w_1=x_2 \vee z_1 \dots w_k = x_{k+1} \vee z_k$, … . The elements $w_0, w_1, \dots, v$ with the elements of the principal ideal $(v]$ form a sublattice C.

Let B^* be the subdirect product of two copies of $D=1+\omega^*$, containing all $(a, b) \in DXD$ for which $a \leqq b$. Then the elements $w_0'=(0, 0)$, $w_1'=(-1, -1), \dots, v'=(-\infty, -\infty)$ form a sublattice of B^* isomorphic to D. We denote the element $(-\infty, 0)$ by y'.

Let us take the principal ideal $(y]$ of L. Then v is a dual atom of this ideal. If B^+ is the extension of $(y]$ by 2^ω then the quotient y/v is isomorphic to $1+\omega^*$.

The ideal y'/v' of B^* is isomorphic to the dual ideal $[y/v)$ of B^+. Let B be the Hall—Dilworth lattice obtained by gluing together B^* and B^+ identifying the corresponding elements under the isomorphism $y'/v' \cong y/v$. (See Fig. 1.) In B^+ the extension $(v][2^\omega]$ is the principal ideal generated by v. This principal ideal with the elements $w_0', w_1', \dots$ form a sublattice C' of B isomorphic to $C (\subseteqq A)$.

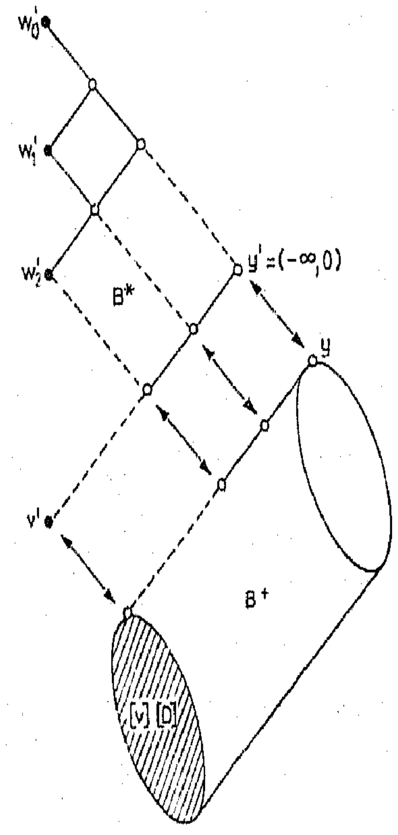

Fig. 1

We have two lattices A and B with isomorphic sublattices $C\cong C'$ (we can assume that A and B are disjoint.) Then the set $M=A\cup B$ ($\cup$ denotes the set-theoretical join) with C and C' identified can be made into a poset by defining $a\leqq b$ if and only if one of the following conditions is satisfied:

(i) $a\leqq b$ in A;
(ii) $a\leqq b$ in B;
(iii) $a\leqq c_1$ in A, $c_2\leqq b$ in B where c_1 and c_2 are corresponding elements under the isomorphism $C\cong C'$;
(iv) $a\leqq c_1$ in B and $c_2\leqq b$ in A where c_1, c_2 are corresponding elements.

We will prove that M is a modular lattice, the lattice obtained by gluing together A and B by $C\cong C'$ as illustrated in Fig. 2.

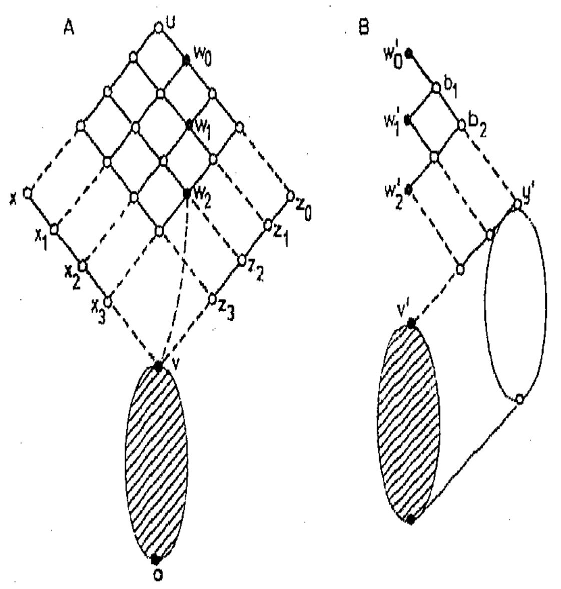

Fig. 2

LEMMA 2. *M is a modular lattice, and L is an epimorphic image of M.*

PROOF. First we prove that M is a lattice, i.e. every two elements $a, b\in M$ have a least upper bound and a greatest lower bound. We can assume that $a\in A$, $b\in B$ and $a, b\notin C$.

If t is an arbitrary element of A such that $t\leqq w_0$ then it is easy to show that there exists a least upper bound $\bar t\in C$. Similarly $\underline{t}$ is the greatest lower bound which exists for every $t\in A$. For the sublattice C' of B, $\bar t$ and $\underline{t}$ exist for every $t\in B$.

If $\bar a, \bar b\in\{w_0, w_1, \dots\}$ then $\bar a\geqq\bar b$ or conversely. Les us assume that $\bar a\geqq\bar b$. Then $a, \bar b\in A$, and if $a\vee\bar b$ denotes the join of a and $\bar b$ in A then this element is obviously the supremum of a and b in M. If $\bar a$ does not exist, then $a\vee\bar b$ is the least upper bound of a and b. Similarly we can prove the existence of $a\vee b$ if $\bar a\in\{w_0, w_1, \dots,\}$, $\bar b\leqq v'$. In the case $a\leqq v$, $b\leqq v'$ the join of a and b is the same as in $(v]$. Similarly we can prove the existence of $a\wedge b$.

It can be easily shown that the lattices A and B have the following two properties:

(*) if $a, b\in A$ (resp. $a, b\in B$) and $a\geqq b, \bar a=\bar b, \underline{a}=\underline{b}$ then $a=b$.

(**) if $c\in C$, $u\notin C$, $u\in A$ (or $u\in B$) then $u\not\geqq c$ impies $u\vee c\notin C$ $\underline{u\vee c}=c$ and similarly $u\not\geqq c>w_i$ implies $u\wedge c\notin C$, $\overline{u\wedge c}=c$.

Let us assume that M is not modular, i.e. that M contains a pentagon with the elements $0<s<r<i$, $0<t<i$. We distinguish several cases. A and B are modular lattices, hence, $r, s, t\in A$ and $r, s, t\in B$ are impossible.

(a) $r\in A$, $r\notin B$, $s, t\in B$ (or symmetrically changing A and B). Then $r\in A$, $s\in B$, $r>s$ imply the existence of a $c_1\in C$ such that $s\leqq c_1<r$. Since $i=s\vee t\in B$, we have a $c_2\in C$ such that $r<c_2\leqq i$. Set $t'=c_2\wedge t$, then $t'\vee s=(c_2\wedge t)\vee s=c_2\wedge(t\vee s)=c_2$. B is modular therefore $c_1=s$. Then $c_1\vee t'=c$ and condition (**) yields $t'\geqq c_2$, i.e. $c_2\leqq t$ which is impossible.

(b) $r, s\in A$, $t\in B$, $r, s, t\notin C$ (or conversely $r, s\in B$, $t\in A$). We have three different possibilities.

(b_1) $0, i\in B$. Then $i\in B$ implies $i=\bar r\vee t=\bar s\vee t$. Similarly $0\in B$ implies $0=\underline{r}\wedge\underline{t}=\underline{s}\wedge t$. By (*) we can assume $\bar r>\bar s$. Then $\bar s\vee(\bar r\wedge t)=\bar r\wedge(\bar s\vee t)=\bar r$ hence by (**) we get $\bar r=\bar r\wedge t$, i.e. $\bar r\leqq t$, a contradiction.

(b_2) $0\in A$, $i\in B$ then $i=\bar r\vee t=\bar s\vee t$, $0=r\wedge t=s\wedge t$. Let $t'=t\wedge\bar r$ then by (**) $\bar t'=\bar r$, i.e. $r\vee t'=s\vee t'=\bar r$. Hence we have a pentagon $0, s, r, t', \bar r$ satisfying $0, s, r, \bar r\in A$, $t'\in B$. Therefore we have to consider the following situation.

(b_3) $0, i\in A$. Then $i=r\vee\bar t=s\vee\bar t$ and $0=r\wedge\underline{t}=s\wedge\underline{t}$. By the modularity of A we get $r\wedge\bar t>s\wedge\bar t$ and by (**) we get that $0, t, s\wedge\bar t, r\wedge\bar t, \bar t$ is a pentagon. Let $r_1=(r\wedge\bar t)\ \underline{t}$, $s_1(s\wedge\underline{t})\ \underline{t}$ then the same argument yields that $\underline{t}, r_1, s_1, t, \bar t$ is a pentagon. This satisfies condition (b).

Let θ be a congruence relation of $1+\omega^*$ defined by $x\equiv y$, $x\neq y$ iff $x, y\in\omega^*$. Then A has a congruence relation θ_1 such that for every prime quotient a/b of L the restriction of θ_1 to the quotient $a//b$ (which denotes the quotient of A corresponding to a/b) is the image of θ by the isomorphism $\varepsilon_{ab}:\underset{\sim}{2}^{\omega}\to a//b$. That implies that the restriction of θ_1 to the subchain $\{w_0, w_1, \dots, v\}$ contains two classes $\{w_0, w_1, \dots\}$ and $\{v\}$.

Let Φ_1 be a congruence relation of B^* defined by: $x\equiv y(\Phi_1)$, $x\neq y$ if and only if either $x, y>w_i'$ for some i or $v'<x, y\leqq y'$. Finally we have a congruence relation Φ_2 on B^+ such that for every prime quotient a/b of $(y]$ the restriction of Φ_2 to $a//b$ is the image of θ by ε_{ab}. The transitive extension $\theta_2\in\mathrm{Con}(B)$ of $\Phi_1\cup\Phi_2$ has the following properties: (1) The restriction of θ_2 to the subchain $\{w_0', w_1', \dots, v'\}$ contains two classes, $\{w_0', w_1', \dots\}$ and $\{v'\}$. (2) The restriction of θ to $(v']$ is the restriction of Φ_2 to $(v]$. Let $\bar\theta\in\mathrm{Con}(M)$ be the transitive extension of θ_1 and θ_2 to M. Then $\bar\theta|A=\theta_1$, $\bar\theta|B=\theta_2$, $A/\theta_1\cong L'$, and B/θ_2 is the lattice $(y]$ adjoined with a new unit element u. Thus we get that $M/\bar\theta$ is isomorphic to L which proves Lemma 2.

The lattice M has the sublattice illustrated on Fig. 3.

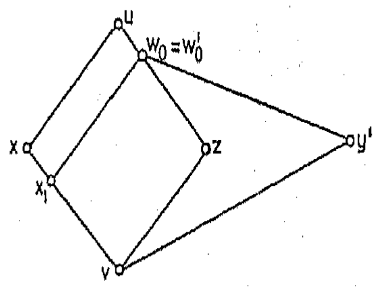

Fig. 3

Let us assume that there is a projectivity $\pi\colon z/v \to x/v$ in L'. We have further in L the projectivity

$$\varrho\colon x/v \nearrow u/z \searrow z/v.$$

The product $\pi \circ \varrho$ of this two functions is a projectivity $x/v \approx x/v$. This projectivity has no inverse in M, since the inverse of ϱ is the projectivity

$$\varrho'\colon x_1/v \nearrow w_0/y' \searrow z/v,$$

and $x_1 < x$. Applying Lemma 1 we get that L cannot be finitely **M**-projected. This contradiction proves our Theorem.

5. Proof of Theorem 2

It is easy to show that if L is a breadth two lattice then the lattice M constructed above is of breadth two. Hence Theorem 1 remains valid in the variety $\mathbf{B}_2$. Let L be a finitely $\mathbf{B}_2$-projected lattice.

Let (v, x, y, z, u) be an isometric diamond where y is meet-irreducible. $t \vee v = u$, $t \neq u$ would imply that $\{v \wedge t, x \wedge t, y \wedge t, x, y, u, v\}$ form an eight element Boolean algebra which is a contradiction to the assumption that L is of breadth two. We have therefore that $t \vee v = u$ implies $t = u$. By Theorem 1 we get that in $L' = L \backslash y^*$ the quotients x/v and z/v are not projective.

Suppose now that L contains a sublattice, say (v, x, y, z, t, u), isomorphic to M_4 with v and u as bottom and top elements, respectively. We can assume that this is an isometric sublattice and L does not contain an other sublattice (v', x', y', z', t', u') isomorphic to M_4 such that $u \leqq v'$, i.e. the given sublattice is "maximal". We prove that one of the elements x, y, z, t is meet-irreducible. If x, y, z, t are all reducible, then we have elements $a \succ x$, $b \succ y$, $c \succ z$, $d \succ t$ and all these elements are different from u. Set $x' = a \vee u$, $y' = b \vee u$, $z' = c \vee u$, $t' = d \vee u$. Then $x' \succ u$, $y' \succ u$, $z' \succ u$ and $t' \succ u$. L is of breadth two hence x', y', z', t' are different and $(u, x', y', z', t', x' \vee y')$ form a sublattice isomorphic to M_4. This contradiction proves that one of the elements is meet-irreducible.

Let $D_0 = (v_0, x_0, y_0, z_0, u_0)$ and $D_1(v_1, x_1, y_1, z_1, u_1)$ be diamonds such that $D_0 \nearrow D_1$, where u_0/z_0 and x_1/v are transposed intervals. We prove that if $x_0, y_0, z_0, x_1, y_1, z_1$ are all meet-reducible elements, then we can change D_0 to D_0', D_1 to D_1' such that $D_0' \nearrow D_1'$ and the bottom of D_i' is the top of D_i.

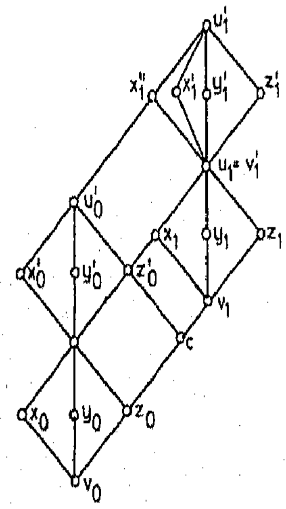

Fig. 4

Indeed, if x_0, y_0 are reducible then we have two elements a, b and $a \succ x_0$, $b \succ y_0$, $a \neq u_0$, $b \neq u_0$. We choose an element c, $z_0 \prec c \leqq z_1$. Then $x_0' = a \vee u_0$, $y_0' = b \vee u_0$, $z_0' = c \vee u_0$ generate a diamond D_0', where $v_0' = u_0$ and $u_0' = x_0' \vee y_0'$. Similarly if x_1, y_1, z_1 are meet-irreducible we get a diamond $D_1' = (v_1', x_1', y_1', z_1', u_1')$ and $v_1' = u_1$. (See Fig. 4.)

Let $x_1'' = u_1 \vee u_0'$. Then $x_1'' \succ u_1$. Since L is of breadth two, $x_1'' < u_1'$, i.e. $(v_1', x_1'', y_1', z_1', u_1') = D_1'$ is a diamond and $D_0' \nearrow D_1'$. Using this procedure we get from a given diamond circle $D_0, D_1, \ldots, D_{n-1}$ in finite steps an other diamond circle which contains a diamond with a meet-irreducible element.

If L contains a sublattice isomorphic to M_4 or a diamond circle then we can assume that this M_4 resp. the diamonds in the circle are isometric. In both cases we have an isometric diamond (v, x, y, z, u) where y is meet-irreducible and x/v, z/v are projective quotients in the sublattice $L' = L \backslash y^*$ which is a contradiction to the assumption that L is finitely $\mathbf{B}_2$-projected.

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SEMICENTRALIZING AUTOMORPHISMS OF PRIME RINGS

By

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Let R be an associative ring and T an automorphism of R. T is called a commuting automorphism of R if $xT(x)=T(x)x$ for all x in R, and it is called a semicommuting automorphism if either $xT(x)=T(x)x$ or $xT(x)=-T(x)x$ is satisfied for each x in R. In [1] L. O. Chung and J. Luh have shown that every semicommuting automorphism of a prime ring is commuting provided that R has either characteristic different from 3 or non-zero centre and thus they proved the commutativity of prime rings having nontrivial semicommuting automorphism except in the indicated cases.

On the other hand, in [3] J. H. Mayne has called T centralizing if $xT(x)-T(x)x\in Z$ for all $x\in R$, where Z is the centre of R, and he has proved that any prime ring with a nontrivial centralizing automorphism is commutative. Now, call T semicentralizing if either $xT(x)-T(x)x$ or $xT(x)+T(x)x$ is in Z for all $x\in R$. Then, we shall show that every semicentralizing (hence every semicommuting) automorphism of a prime ring is in fact commuting.

As to notations, we set $[x, y]=xy-yx$, $(x, y)=xy+yx$ for $x, y\in R$; and for the sake of simplicity, we shall write $T(x)=x'$, $R_+=\{x\in R|xx'-x'x\in Z\}$, $R_-=\{x\in R|xx'+x'x\in Z\}$.

We need two lemmas.

Lemma 1. *Let R be any ring and T a semicentralizing automorphism of R. If x and y are in R_+ (resp. in R_-), then $x+y$ is in R_+ (resp. in R_-) if and only if $x-y$ is in R_+ (resp. in R_-).*

Proof. Consider $x, y\in R_+$. Then the following implications are immediate.

$$[x+y,\ x'+y']\in Z \Leftrightarrow [x, y']+[y, x']\in Z \Leftrightarrow [x, -y']+[-y, x']\in Z \Leftrightarrow [x-y, x'-y']\in Z.$$

The same argument can be repeated by replacing $[\,,]$ by $(\,,)$ for the statement concerning R_-.

Lemma 2. *Let R be a prime ring and T a semicentralizing automorphism of R. If $y\notin R_+$, then $y^2y'^2=0$.*

Proof. In the case of char $(R)=2$ semicentralizing automorphisms are centralizing and thus by [3; Theorem] they are commuting automorphisms. So in the sequel we shall assume that char $(R)\neq 2$. Since $yy'+y'y\in Z$, we get

$$[y', y^2] = [yy'+y'y, y] = 0 = [y'y+yy', y'] = [y, y'^2],$$

which gives that

$$y'y^2 = y^2y' \quad\text{and}\quad y'^2y = yy'^2. \tag{1}$$

Since

$$[y^2+y, y'^2+y'] = [y, y'] = [-y, -y'] = [y^2-y, y'^2-y']$$

is not contained in Z, neither y^2+y nor y^2-y is in R_+. So, y^2+y and y^2-y are in R_- and by Lemma 1

$$(y^2+y)-(y^2-y) = 2y \in R_- \quad \text{implies} \quad (y^2+y)+(y^2-y) = 2y^2 \in R_-,$$

and hence $y^2 \in R_-$. Thus, $y^2y'^2+y'^2y^2=2y^2y'^2 \in Z$, that is, $y^2y'^2 \in Z$.

Since $y^2+y \notin R_+$, using (1) we obtain

$$[[y^2+y]^2, y'^2+y'] = 2[y^3, y'] = 0.$$

The fact that char $(R) \neq 2$ together with (1) implies that $[y^3, y']=y^2[y, y']=0$, so we have $y'^2y^2[y, y']=0$.

Now, y'^2y^2 is an element in the centre of the prime ring R and $[y, y'] \neq 0$, thus $y'^2y^2=0$, as required.

We are ready to prove our theorem.

Theorem. *Let R be a prime ring and T a semicentralizing automorphism of R. Then T is a commuting automorphism.*

Proof. Consider the element xy^2+y^2, where $x \in R$ and $y \notin R_+$. Then

$$(xy^2+y^2)(x'y'^2+y'^2) \pm (x'y'^2+y'^2)(xy^2+y^2) = c \in Z$$

or, using Lemma 2,

$$xy^2x'y'^2+y^2x'y'^2 \pm (x'y'^2xy^2+y'^2xy^2) = c.$$

Postmultiplication by y'^2 gives

$$xy^2x'y'^4+y^2x'y'^4 = cy'^2. \tag{2}$$

Similarly, considering xy^2-y^2 we obtain

$$xy^2x'y'^4-y^2x'y'^4 = c_1y'^2 \tag{3}$$

for some c_1 in Z. If $c \neq c_1$, then from (2) and (3) one obtains $2y^2x'y'^4=(c-c_1)y'^2$, and premultiplicating this by y'^2 gives $(c-c_1)y'^4=0$, that is $y^4=0$. If $c=c_1$, then from (2) and (3) we get $2y^2x'y'^4=0$, that is, $y^2x'y'^4=0$. Since R is prime we obtain $y^4=0$. Thus,

$$\text{if} \quad y \notin R_+, \quad \text{then} \quad y^4 = 0. \tag{4}$$

Next, we shall show that if $y^n=0$ for all $y \notin R_+$ where n is a fixed integer greater than 1, then $y^{n-1}=0$ for all $y \notin R_+$.

In fact, for any $x \in R$ and $y \notin R_+$ consider xy^{n-1}. If $xy^{n-1} \notin R_+$, then

$$(xy^{n-1})^n = 0, \tag{5}$$

if not, then

$$(xy^{n-1}-y)+(xy^{n-1}+y) = 2xy^{n-1} \in R_+$$

and

$$(xy^{n-1}-y)-(xy^{n-1}+y) = -2y \in R_-$$

will imply, by Lemma 1, that either $xy^{n-1}-y \notin R_+$ or $xy^{n-1}+y \notin R_+$. Thus,

$$(xy^{n-1}-y)^n = (xy^{n-1})^n - y(xy^{n-1})^{n-1} + \ldots + (-y)^{n-1}(xy^{n-1}) = 0$$

or

$$(xy^{n-1}+y)^n = (xy^{n-1})^n + y(xy^{n-1})^{n-1} + \ldots + y^{n-1}(xy^{n-1}) = 0.$$

In both cases, multiplying through by xy^{n-1} on the left we obtain $(xy^{n-1})^{n+1}=0$. Combining this and (5) we see that $(xy^{n-1})^{n+1}=0$ for all $x \in R$.

Now, using Lemma 1.1 in [2] we conclude that $y^{n-1}=0$. Consequently, from (4) we see that $R=R_+$, and hence T is a commuting automorphism.

As an immediate consequence of this theorem and Lemma 1 in [1] we obtain the following result generalizing [1; Theorems 1, 2] and [3; Theorem].

Corollary. *A prime ring possessing a nontrivial semicentralizing automorphism is a commutative integral domain. In particular, a prime ring possessing a nontrivial semicommuting automorphism is a commutative integral domain.*

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ON THE SUPERSOLVABILITY OF FINITE GROUPS. I

By
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1. Introduction

It was proved by J. BUCKLEY [1] that if G is a finite group of odd order in which every minimal subgroup is normal then G is supersolvable.

In [2], ROBERT W. VAN DER WAALL showed that this result is a direct corollary of a theorem of DOERK; see [3]. A subgroup H of G is called pronormal in G if for x in G, H is conjugate to H^x in $\langle H, H^x\rangle$. It is known that a subgroup H of G is normal in G iff it is both subnormal and pronormal in G (Exercise 6, p. 14 of [4]). An immediate consequence of this and Doerk's Theorem is the following generalization of Buckley's result.

THEOREM A. *If G is a finite group of odd order n in which every minimal subgroup is pronormal in G, then G is supersolvable.*

A familiar consequence of the supersolvability of G is that G possesses a normal subgroup H of order p, where p is the largest prime in $\Pi(G)$, such that G/H is supersolvable (p. 159 of [5]).

We shall prove the following result.

THEOREM B. *Let G be a finite group. If there exists a normal subgroup H such that G/H is supersolvable, $(|H|, 2)=1$, and every minimal subgroup of H is pronormal in G, then G is supersolvable.*

Theorem B is false if the assumption $(|H|, 2)=1$ is omitted. This is easily seem by examination of $SL(2, 3)$.

The following generalization of Ito's Theorem (Theorem 5.3 b), p. 283 of [7]) follows at once from Theorem B.

COROLLARY. *Let G be a finite group. Let G' be the commutator subgroup of G. If $(|G'|, 2)=1$ and every minimal subgroup of G' is pronormal in G, then G' is nilpotent.*

Our notation is standard. All groups in this paper are finite.

2. Known results

For completeness we state the following known results:

THEOREM 1 (Doerk, p. 721 of [7]). *Let G be a non-supersolvable group, all of whose proper subgroups are supersolvable. Then*

(i) *G contains one and only one normal Sylow subgroup P.*

(ii) *$P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(P)$ and $P/\Phi(P)$ is not cyclic.*

(iii) *P is of exponent p if $p>2$.*

THEOREM 2 (Schur—Zassenhaus, p. 221 of [4]). *Let H be a normal Hall subgroup of G. Then G possesses a subgroup K which is a complement to H in G.*

THEOREM 3 (Exercise 7.2.22, p. 159 of [6]). *If G/H and G/K are supersolvable, then $G/H\wedge K$ is supersolvable.*

THEOREM 4 (p. 173 of [4]). *If $G=H\Phi(G)$ for some subgroup H of G, then $H=G$.*

THEOREM 5 (Tate, p. 431 of [7]; see also [8]). *Let P be a Sylow p-subgroup of G. Let H be a normal subgroup of G. If $H\wedge P\leqq\Phi(P)$, then H possesses a normal p-complement.*

THEOREM 6 (Huppert, p. 713 of [7]). *G is supersolvable iff $G/\Phi(G)$ is supersolvable.*

3. Proof of Theorem B

Suppose the theorem false and let G be a counterexample of minimal order. Let K be any proper subgroup of G. If $K\leqq H$, then K is supersolvable by Theorem *A*. If $K\nleqq H$, set $G_1=KH$. If $G_1<G$, then G_1 is supersolvable by our choice of G. Since G_1 is supersolvable, so is K. Hence we may assume that $G_1=G=KH$. Clearly $G/H\cong K/K\wedge H$. It is immediate that K satisfies the conditions of the theorem. Hence K is supersolvable by our choice of G. Therefore all proper subgroups of G are supersolvable. By Theorem 1 (i), G then contains precisely one normal Sylow subgroup P. Set $H_1=H\wedge P$. Clearly $H_1\lhd G$. We argue that $H_1\neq 1$. By Theorem 2, G possesses a subgroup K which is a complement to P in G. Since $G/P\cong K$, we conclude that G/P is supersolvable. Theorem 3 tells us that $G/H\wedge P$ is supersolvable. Hence if $H_1=H\wedge P=1$, then G would be supersolvable, a contradiction. Thus $H_1\neq 1$. Theorem 1 (ii) implies that $H_1\Phi(P)=P$ or $H_1\Phi(P)=\Phi(P)$. If $H_1\Phi(P)=P$, then $H_1=P$ by Theorem 4. Since P is of odd order, then $P=\Omega_1(P)=\langle y|y^p=1, y\in P\rangle$ by Theorem 1 (iii). Obviously every minimal subgroup of P is subnormal in G. Since $P\leqq H$, every minimal subgroup of P is then pronormal in G. Now Exercise 6, p. 14 of [4] implies that every minimal subgroup of P is normal in G and consequently $P=\Omega_1(P)=Z(P)$, i. e. P is an elementary abelian p-group. Hence P is the minimal normal subgroup of G and consequently $|P|=p$, a contradiction. Hence we may assume that $H_1\Phi(P)=\Phi(P)$. Theorem 5 implies that H possesses a normal p-complement, i.e. $H=H_1L$, L is a normal subgroup of H and $(|H_1|, |L|)=1$. We argue that $L\neq 1$. Suppose false. Then $H=H_1\leqq\Phi(P)$. Since $G/H/\Phi(P)/H\cong G/\Phi(P)$, and G/H is supersolvable, then $G/\Phi(P)$ is supersolvable. Since $\Phi(P)\leqq\Phi(G)$ and $G/\Phi(P)$ is supersolvable then $G/\Phi(G)$ is supersolvable. Now Theorem 6 implies that G is supersolvable, a contradiction. Thus $L\neq 1$. Since L char H and $H\lhd G$, then $L\lhd G$. Set $\bar{G}=G/L$ and let $\bar{H}$ be the image of H in $\bar{G}$. Clearly $\bar{H}\cong H_1\neq 1$. Let $\bar{N}$ be any subgroup of order p in $\bar{H}$. Let N be the inverse image of $\bar{N}$ in H. Then$=ML$, where M is a subgroup of order p in H. Hence M is pronormal in G. Obviously M is subnormal in G. Applying Exercise 6, p. 14 of [4] once again, it follows that $M\lhd G$ and consequently $\bar{N}\lhd\bar{G}$. Since $\bar{G}/\bar{H}\cong G/H$ and every minimal subgroup of $\bar{H}$ is normal in $\bar{G}$, then $\bar{G}=G/L$ supersolvable by our choice of G. Now Theorem 3 implies that $G/P\wedge L=G$ is supersolvable, contrary to our choice of G. Theorem B is thus proved.

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A REMARK ON STRICT RADICAL CLASSES OF GROUPS

By

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A radical class is *strict* if the radical of every object A contains every radical subobject of A or, equivalently, if all subobjects of semisimple objects are semi-simple. Strict radical classes were first discussed by KUROSH [2], in connection with groups. Subsequently, TSALENKO [4] has noted that the collection of strict radical classes of groups is a proper class, so that, in a sense, such radical classes are plentiful. It turns out, however, that they are never hereditary.

THEOREM. *Let $\mathcal{R}\neq\{e\}$ be a hereditary radical class of groups. The following conditions are equivalent.*

(i) *$\mathcal{R}$ is strict;*
(ii) *$\mathcal{R}$ is closed under formation of free products;*
(iii) *$\mathcal{R}$ is the class of all groups.*

PROOF. (i)⇒(ii): If $G, H\in\mathcal{R}$ then (denoting their free product by G^*H) we have $G, H\subseteqq\mathcal{R}(G^*H)$, so G^*H is in $\mathcal{R}$.

(ii)⇒(iii): Let α be any cardinal number. Then $\mathcal{R}$ contains groups G, H, with $|G|, |H|\geqq\alpha$. Let $[G, H]$ be the subgroup of G^*H generated by $\{ghg^{-1}h^{-1}\mid g\in G, h\in H\}$. Then $[G, H]$ is freely generated by $\{ghg^{-1}h^{-1}\mid g\in G, h\in H\}$ and $[G, H]\triangleleft G^*H$ (see [1], p. 196, Problems 23, 24). Since $G^*H\in\mathcal{R}$ and $\mathcal{R}$ is hereditary, $\mathcal{R}$ contains a free group of rank $\geqq\alpha^2\geqq\alpha$, and therefore all groups of cardinality $\leqq\alpha$.

Obviously (iii)⇒(i). Q.e.d.

We note, by way of contrast, that there are highly non-trivial hereditary strict radicals of associative rings, e.g. the generalized nil radical and the upper radicals defined by semi-simple radical classes (though STEWART [3] has proved that the only *strongly* hereditary strict radical properties in this context are defined in terms of additive groups of rings).

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ESSENTIALLY CLOSED CLASSES OF RINGS AND UPPER RADICALS

By

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In [6] HEYMAN and ROOS defined the essential cover $\mathcal{M}_k$ of a class $\mathcal{M}$ to be the class of all essential extensions of rings belonging to $\mathcal{M}$. $\mathcal{M}$ is called essentially closed if $\mathcal{M}_k=\mathcal{M}$. Every class $\mathcal{M}$ has a unique essential closure, i.e. a smallest essentially closed class containing $\mathcal{M}$. If $\mathcal{M}$ is a hereditary class of (semi) prime rings it was shown to be (weakly) special if and only if $\mathcal{M}$ is essentially closed. In addition, $\mathcal{M}_k$ is the smallest (weakly) special class containing $\mathcal{M}$. Further, it was shown that the upper radical $\mathcal{U}\mathcal{M}$ determined by $\mathcal{M}$ is hereditary if and only if $\mathcal{U}\mathcal{M}$ has the intersection property with respect to $\mathcal{M}_k$.

These results were generalized by ANDERSON and WIEGANDT in [1]. They showed that in the context of weakly special classes it is sufficient to require only regularity instead of hereditariness. Moreover, their results were valid for alternative as well as associative rings.

In this paper we generalize some of the results of Anderson and Wiegandt. Moreover we also extend most of the results obtained in [1] and [6] to classes containing all zero rings. All rings to be considered will be associative and the major knowledge of radical theory required for our purposes is contained in [4] and [8]. For a ring R write $I\lhd R$ and $I\lhd\cdot R$ to mean I is an ideal of R and I is an essential ideal of R respectively.

We will be needing the following well-known results.

PROPOSITION 1. *If $0\neq B\lhd R$ and B is not essential in R, then there exists an ideal C of R such that R/C contains an essential ideal isomorphic to B.* [5]

PROPOSITION 2. *If $I\lhd R$ where I is a semiprime ring, then $I^*=I_r=I_l$ where I^*, I_r and I_l denote the annihilator, right annihilator and the left annihilator of I in R respectively, and $I\cap I^*=0$.* [2]

COROLLARY 1. *If $I\lhd R$ where I is a semiprime ring, then we can show in the same way as in Proposition 1, that*

$$(I+I^*)/I^* \simeq I/(I\cap I^*) \simeq I\lhd\cdot R/I^*.$$

PROPOSITION 3. *Let $\mathcal{M}$ be a class of semiprime rings. If $C\lhd B\lhd R$ such that $B/C\in\mathcal{M}$, then $C\lhd R$.* [2]

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PROPOSITION 4. *A ring K maps homomorphically onto a nonzero subdirectly irreducible ring with nilpotent core if and only if there exists and ideal C in K such that $C^2 \neq C$.* [2]

PROPOSITION 5. *Let $\mathcal{M}$ be an arbitrary weakly special class. Then the radical $S_{\mathcal{M}}$ is hypernilpotent and for an arbitrary A the radical $S_{\mathcal{M}}(A)$ coincides with the intersection of all ideals T_α of A such that $A_\alpha = A/T_\alpha$ is a nonzero ring from the class $\mathcal{M}$. Conversely, every hypernilpotent radical is the upper radical defined for some weakly special class of rings.* [7]

The following definitions will be needed.

DEFINITION 1. A class $\mathcal{M}$ of rings is said to be left regular if every nonzero left ideal of a ring of $\mathcal{M}$ has a nonzero homomorphic image in $\mathcal{M}$.

DEFINITION 2. A class $\mathcal{M}$ of rings is called an α-class if every nonzero ideal of a ring of $\mathcal{M}$ has a nonzero accessible subring which can be mapped onto a nonzero ring in $\mathcal{M}$.

Since a regular class of rings is also an α-class, we can now generalize Theorem 1 [1] of ANDERSON and WIEGANDT for α-classes in part (a) of the next theorem.

THEOREM 1. (a) *If $\mathcal{M}$ is an α-class of rings then $\mathcal{M}_k$ is also an α-class of rings.*

(b) *Let $\mathcal{M}$ be any regular class of rings containing all the zero rings, then $\mathcal{M}_k$ is a regular class containing all the zero rings.*

(c) *If $\mathcal{M}$ is a left regular class of semiprime rings, then $\mathcal{M}_k$ is a left regular class of semiprime rings.*

PROOF. (a) Let $R \in \mathcal{M}_k$ and $0 \neq I \lhd R$. There exists $E \in \mathcal{M}$ such that $E \lhd \cdot R$. Hence $0 \neq I \cap E = D$ is an ideal of both I and E. $E \in \mathcal{M}$ implies that there is a nonzero accessible subring D_n of D such that D_n can be mapped onto a nonzero ring in $\mathcal{M}$. Clearly $D_n \lhd \ldots \cdot \lhd D \lhd I$ so that D_n is an accessible subring of I, and hence $\mathcal{M}_k$ is an α-class of rings.

(b) Since $\mathcal{M} \subseteq \mathcal{M}_k$, it is clear that $\mathcal{M}_k$ contains all the zero rings. It suffices to show that $\mathcal{M}_k$ is regular. Let $0 \neq I \lhd R \in \mathcal{M}_k$. If I has a nonzero ideal J with $J \neq J^2$, then by Proposition 4, I can be mapped onto a subdirectly irreducible ring, say I/P, with a heart H/P which is a zero ring. Furthermore with $H/P \in \mathcal{M}$ and H/P essential in I/P, it follows that $I/P \in \mathcal{M}_k$ and $\mathcal{M}_k$ therefore regular. We may therefore assume that I is hereditarily idempotent. There exists an essential ideal L in R, $L \in \mathcal{M}$ and $I \cap L \neq 0$. Since $I \cap L \lhd L \in \mathcal{M}$, there exists a homomorphic image $(I \cap L)/K \neq 0$ of $I \cap L$ which is in $\mathcal{M}$. Furthermore, $I \cap L$ being an ideal of I, is hereditarily idempotent and so $K \lhd I$. Hence, I can be mapped onto I/K which contains $(I \cap L)/K \in \mathcal{M}$ as an ideal. If $(I \cap L)/K$ is essential in I/K, then $I/K \in \mathcal{M}_k$ and we are done. If $(I \cap L)/K$ is not essential in I/K, Proposition 1 implies that there is a homomorphic image of I/K containing an essential ideal isomorphic to $(I \cap L)/K \in \mathcal{M}$. Therefore I has a nonzero homomorphic image in $\mathcal{M}_k$.

(c) It is easily verified that $\mathcal{M}_k$ consists of semiprime rings so that it is sufficient to show that $\mathcal{M}_k$ is left regular. Let I be any nonzero left ideal of $R \in \mathcal{M}_k$. R has an essential ideal $L \in \mathcal{M}$. If $I \cap L = 0$, then $LI = 0$ so that $I \subseteq L_r^*$. From Proposition 2 it follows that $I = 0$ which is against our assumption. So $I \cap L \neq 0$, $I \cap L \lhd I$ and $I \cap L$ is a left ideal of $L \in \mathcal{M}$. Since $\mathcal{M}$ is a left regular class of rings there exists $K \lhd I \cap L$ such that $0 \neq (I \cap L)/K \in \mathcal{M}$. Proposition 3 implies that $K \lhd I$. Then $0 \neq (I \cap L)/K \lhd I/K$. We can now complete the proof in a similar way as in (b) of the theorem.

The following theorem gives conditions on the class $\mathcal{M}$ for $\mathcal{M}_k = (\mathcal{M}_k)_k$.

THEOREM 2. (a) *If $\mathcal{M}$ is a class of idempotent rings, then $\mathcal{M}_k = (\mathcal{M}_k)_k$.*

(b) *If $\mathcal{M}$ is any class containing all the zero rings and if $\mathcal{M}$ satisfies the extension property, then $\mathcal{M}_k = (\mathcal{M}_k)_k$.*

PROOF. (a) We always have that $\mathcal{M}_k \subseteq (\mathcal{M}_k)_k$. Suppose $\mathcal{M}_k \subsetneqq (\mathcal{M}_k)_k$. Then there exists a ring $R \in (\mathcal{M}_k)_k$ but $R \notin \mathcal{M}_k$. R has an essential ideal $J \in \mathcal{M}_k$ and $I \lhd \cdot J$ with $I \in \mathcal{M}$. Since I is idempotent it follows that $I \lhd R$. It is easy to see that $I \lhd \cdot R$ which implies $R \in \mathcal{M}_k$, a contradiction. So $\mathcal{M}_k = (\mathcal{M}_k)_k$.

(b) As in (a) we suppose that $\mathcal{M}_k \subsetneqq (\mathcal{M}_k)$ and $R \in (\mathcal{M}_k)_k$ but $R \notin \mathcal{M}_k$. This means that

$$(*) \qquad \text{if } E \lhd \cdot R \text{ then } E \notin \mathcal{M}.$$

There is an ideal I of R such that $I \lhd \cdot R$, $I \in \mathcal{M}_k$, $I \notin \mathcal{M}$. With $I \in \mathcal{M}_k$ we have a $J \lhd \cdot I$ and $J \in \mathcal{M}$. Since $J_R^3 \subseteq J$ where J_R is the ideal generated by J in R, it follows that $(J_R^2 + J)/J$ is a zero ring, $(J_R^2 \neq 0)$ and hence in $\mathcal{M}$. (If $J_R^2 = 0$, it is easily verified that R has an essential ideal, i.e. J_R, in $\mathcal{M}$). With $J \in \mathcal{M}$ and $\mathcal{M}$ satisfying the extension property, we have $J_R^2 + J \in \mathcal{M}$. Following the same argument we can conclude that $J_R \in \mathcal{M}$ if we consider $J_R/(J_R^2 + J)$. It is easily verified that $J_R \lhd \cdot R$ since $J \lhd \cdot I$, $I \lhd \cdot R$ and $J \subseteq J_R$. This however is a contradiction with $(*)$. We conclude that $\mathcal{M}_k = (\mathcal{M}_k)_k$.

In [1], ANDERSON and WIEGANDT ask if the essential cover of a regular class of semi (prime) rings is essentially closed. A partial answer to this question is found in the following theorem.

THEOREM 3. *Let $\mathcal{M}$ be a regular class of semiprime rings such that $\mathcal{M}$ is subdirectly closed. Then $\mathcal{M} = \mathcal{M}_k$ if and only if $\mathcal{M}$ is the semisimple class of a hereditary radical.*

PROOF. The proof follows immediately by using Theorem 3 and Proposition 4 of [1].

COROLLARY 2. *Let $\mathcal{M}$ be a regular class of semiprime rings such that $\mathcal{M}_k$ is subdirectly closed. Then $\mathcal{M}_k = (\mathcal{M}_k)_k$ if and only if $\mathcal{M}_k$ is the semisimple class of a hereditary radical.*

The following example illustrates that a class $\mathcal{Q}$ of rings need not be regular to have $\mathcal{Q} = \mathcal{Q}_k$.

EXAMPLE. Let $\mathcal{Q}$ be the class of all quasi-semiprime rings, (cf. [3]), that is rings R for which $RAR = 0$ where $A \lhd R$, implies $A = 0$. Let now $R \in \mathcal{Q}$ and $R \lhd \cdot S$. Suppose that $SAS = 0$ where $A \lhd S$. Then $R(A \cap R)R \subseteq S(A \cap R)S \subseteq SAS = 0$ and since $R \in \mathcal{Q}$, $A \cap R = 0$. Then, because of the essentiality of R in S, $A = 0$ so that $S \in \mathcal{Q}$. Hence $\mathcal{Q} = \mathcal{Q}_k$. It was however shown in [3] that

$$\mathcal{U}\mathcal{Q} = \{R \mid R \text{ cannot be mapped homomorphically onto a nonzero ring in } \mathcal{Q}\}$$

is not a radical class, so that $\mathcal{Q}$ cannot be regular.

In order to extend and sharpen the results of Theorem 2 in [1], we need the following three theorems.

THEOREM 4. *If $\mathcal{M}$ is a regular class of rings containing no or all nonzero zero rings, then $\mathcal{U}\mathcal{M}_k$ is hereditary.*

PROOF. If $\mathcal{M}$ contains no nonzero zero rings we can regard $\mathcal{M}$ as a class of semiprime rings. For if $0\neq I\triangleleft R\in\mathcal{M}$ and $I^n=0$ with $I^{n-1}\neq 0$, then I^{n-1} is a nonzero zero ring, which, in view of the fact that $\mathcal{M}$ is regular, is a contradiction. We now show that in this case $\mathcal{U}\mathcal{M}_k$ is hereditary. If not, there exists a ring $R\in\mathcal{U}\mathcal{M}_k$ containing a nonzero ideal $A\notin\mathcal{U}\mathcal{M}_k$. Then A has some nonzero image $A/I\in\mathcal{M}_k$ and since A/I is a semiprime ring, $A/I\triangleleft R/I$. We may therefore, without loss of generality, assume that $A\in\mathcal{M}_k$. Then there exists $0\neq I\triangleleft\cdot A$ such that $I\in\mathcal{M}$. Considering I_R, we know that $I_R^3\triangleleft R$. Let K be any ideal of R such that $I_R^3\triangleleft K=0$. Now we have $(I_R\cap K)^3\subseteqq I_R^3\cap K=0$, and hence $I_R\cap K$ is nilpotent. Since $\mathcal{M}$ is regular, by Theorem 1 (b) so is $\mathcal{M}_k$ and therefore $I_R\cap K\triangleleft A\in\mathcal{M}_k$ yields $I_R\cap K=0$. Thus by $I\cap K\subseteqq I_R\cap K=0$ and $I\triangleleft\cdot A$ it follows that $K=0$ and hence $I_R^3\triangleleft\cdot R$. If $I_R^3\in\mathcal{M}$, we have $R\in\mathcal{M}_k$, a contradiction. If $I_R^3\notin\mathcal{M}$, it has an image $0\neq I_R^3/P\in\mathcal{M}$. Once again, since I_R^3/P is a semiprime ring, we have $I_R^3/P\triangleleft R/P$. If $I_R^3/P\triangleleft\cdot R/P$, we have $R/P\in\mathcal{M}_k$, a contradiction and if I_R^3/P is not essential in R/P, we can find a homomorphic image $(R/P)/(K/P)\cong R/K$ of R containing an isomorphic copy of I_R^3/P as essential ideal. Then $R/K\in\mathcal{M}_k$ which once again is a contradiction. So $\mathcal{U}\mathcal{M}_k$ is hereditary indeed.

Suppose then that $\mathcal{M}$ contains all zero rings and that $\mathcal{U}\mathcal{M}_k$ is not hereditary. Then $A\triangleleft R\in\mathcal{U}\mathcal{M}_k$ where $A\notin\mathcal{U}\mathcal{M}_k$ so that there exists $L\triangleleft A$ such that $A/L\in\mathcal{M}_k$. In addition assume that an arbitrary homomorphic image A' of A contains a nilpotent essential ideal. Then A has an ideal $I\neq I^2$, for otherwise A would be hereditarily idempotent and all its images would be hereditarily idempotent. Thus $I_R\neq I_R^2$, so by Proposition 5 R can be mapped homomorphically onto a subdirectly irreducible ring R/Q with heart H/Q, which is a zero ring. Since $H/Q\triangleleft\cdot R/Q$ it follows that $R/Q\in\mathcal{M}_k$. But $R\in\mathcal{U}\mathcal{M}_k$ which is homomorphically closed. Thus no homomorphic image of A can have a nilpotent essential ideal. We conclude then that A/L is a semiprime ring for if P/L is a nilpotent ideal of A/L there exists a homomorphic image $(A/L)/(K/L)\simeq A/K$ of A/L containing an isomorphic copy of P/L as essential ideal, and thus a homomorphic image of A containing a nilpotent essential ideal, a contradiction.

Since $L\triangleleft A\triangleleft R$ and A/L is semiprime we have that $L\triangleleft R$ and $A/L\triangleleft R/L$, so again we may assume that $A\in\mathcal{M}_k$. By assumption, A then has an essential ideal $J\in\mathcal{M}$. We then have $J_R^3\subseteqq J\subseteqq J_R$ and let us assume the last containment is proper. Then $J_R/J_R^2\neq 0$ and thus is a zero ring in $\mathcal{R}$ and is an ideal of R/J_R^2. If J_R/J_R^2 is essential in R/J_R^2 then $R/J_R^2\in\mathcal{M}_k$, a contradiction, for $R\in\mathcal{U}\mathcal{M}_k$ which is homomorphically closed. If it is not essential then there exists a homomorphic image of R/J_R^2 containing as essential ideal an isomorphic copy of J_R/J_R^2 and hence this image is again in $\mathcal{M}_k$ contradicting $R\in\mathcal{U}\mathcal{M}_k$. Thus assuming $J\subseteqq J_R$ to be a proper containment is false and so $J\triangleleft R$. Now $J\in\mathcal{M}$. If $J\triangleleft\cdot R$ then again $R\in\mathcal{M}_k$, a contradiction. If J is not essential in R there is again a homomorphic image of R containing as essential ideal an isomorphic copy of J, placing this image in $\mathcal{M}_k$ and hence a homomorphic image of R in $\mathcal{M}_k$, contradicting $R\in\mathcal{U}\mathcal{M}_k$. Thus $\mathcal{U}\mathcal{M}_k$ is hereditary, completing the proof.

THEOREM 5. *Let $\mathcal{M}$ be a regular class of rings containing no or all nonzero zero rings. Then $\mathcal{U}\mathcal{M}_k=\mathcal{U}\mathcal{M}$ if and only if $\mathcal{U}\mathcal{M}$ is hereditary.*

PROOF. One way is clear since $\mathcal{U}\mathcal{M}_k$ is always hereditary. Now suppose that $\mathcal{U}\mathcal{M}$ is hereditary and let $R\in\mathcal{M}_k$. By definition of $\mathcal{M}_k$, R has an essential ideal $I\in\mathcal{M}$. Therefore $I\in S\mathcal{U}\mathcal{M}$ and since $\mathcal{U}\mathcal{M}$ is hereditary, $\mathcal{U}\mathcal{M}(I)=I\cap\mathcal{U}\mathcal{M}(R)$. But $\mathcal{U}\mathcal{M}(I)=0$ hence $I\cap\mathcal{U}\mathcal{M}(R)=0$, and since $I\triangleleft\cdot R$, we have that $\mathcal{U}\mathcal{M}(R)=0$, so that $R\in S\mathcal{U}\mathcal{M}$. Hence $\mathcal{M}_k\subseteqq S\mathcal{U}\mathcal{M}$, so that $\mathcal{U}\mathcal{M}\subseteqq\mathcal{U}\mathcal{M}_k$. The reverse inclusion is always true, hence $\mathcal{U}\mathcal{M}=\mathcal{U}\mathcal{M}_k$ which was to be proved.

It is interesting to remark that the previous two theorems imply $\mathcal{U}\mathcal{M}_k=\mathcal{U}[(\mathcal{M}_k)_k]=\ldots=\mathcal{U}\mathcal{M}^c$ provided $\mathcal{U}\mathcal{M}_k$ is either supernilpotent or subidempotent.

THEOREM 6. *Let $\mathcal{M}$ be a regular class of rings containing no or all nonzero zero rings. Then for every ring R, $\mathcal{U}\mathcal{M}_k(R)=\cap Q_\alpha$ where $R/Q_\alpha\in\mathcal{M}_k$.*

PROOF. If $\mathcal{M}$ does not contain nonzero zero rings, then as in Theorem 4 one can show that $\mathcal{M}$ is a class of semiprime rings. Then for any ring R, $\mathcal{U}\mathcal{M}_k(R)=\cap I_\alpha$ where $R/I_\alpha\in\mathcal{M}_k$, the proof of which runs the same as the one given for Proposition 5 in [7].

Suppose then that $\mathcal{M}$ contains all zero rings and let $\{Q_\alpha\}$ be the set of all ideals of an arbitrary ring R such that $R/Q_\alpha\in\mathcal{M}_k$. It is clear that $\mathcal{U}\mathcal{M}_k(R)\subseteqq\cap Q_\alpha$ always holds, so suppose $\mathcal{U}\mathcal{M}_k(R)\subset\cap Q_\alpha=J$. The hereditariness of $\mathcal{U}\mathcal{M}_k$ then implies that $\mathcal{U}\mathcal{M}_k(J)=J\cap\mathcal{U}\mathcal{M}_k(R)\subset J$ so that J is not $\mathcal{U}\mathcal{M}_k$-radical. Therefore it can be mapped homomorphically onto a nonzero ring in $\mathcal{M}_k$. If J has an ideal $P\neq P^2$ then also $P_R\neq P_R^2$. Furthermore $P_R\subseteqq J$ and let us now choose an element $p\in P_R$ but $p\notin P_R^2$. We then construct the ideal U maximal relative to $P_R^2\subseteqq U$, $p\notin U$. Following the proof of Proposition 4 in [2] we conclude that $R/U\in\mathcal{M}_k$, so U must be one of the Q_α's. Since $J=\cap Q_\alpha$ it follows that $J\subseteqq U$. However $p\notin U$ whereas $p\in J$. This contradiction implies that J is hereditarily idempotent and hence cannot be mapped homomorphically onto a ring having a zero ring as essential ideal. But then the nonzero map J/I say, which J has to have in $\mathcal{M}_k$, is semiprime (being hereditarily idempotent). Hence by Corollary 1, we have

$$[J/I+(J/I)^*]/(J/I)^*\cong J/I/[(J/I)\cap(J/I)^*]\cong J/I\triangleleft\cdot R/I/(J/I)^*.$$

But J/I has an essential ideal $K/I\in\mathcal{M}$ and, being hereditarily idempotent, $K/I\triangleleft(R/I)/(J/I)^*$. Since K/I is essential in J/I, it is also essential in $(R/I)/(J/I)^*$ so that $(R/I)/(J/I)^*\in\mathcal{M}_k$.

We now construct in a similar way as in Proposition 1 [7], the ideal $Q=\{a\in R\mid aJ\subseteqq I \text{ and } Ja\subseteqq I\}$. With results available from [7] we have that $Q\triangleleft R$, $Q\cap J=I$ and $R/Q\cong(R/I)/(J/I)^*$. Then $R/Q\in\mathcal{M}_k$, so Q_α must be one of the Q_α's, hence $J\subseteqq Q$. However $J=\cap Q_\alpha=(\cap Q_\alpha)\cap Q=J\cap Q=I\subset J$, which is a contradiction. The proof is therefore complete.

With the aid of the previous three theorems we can now extend Theorem 7 [6] to allow $\mathcal{M}$ to contain all zero rings and sharpen their results by replacing $\mathcal{M}^c$ with $\mathcal{M}_k$.

THEOREM 7. *Let $\mathcal{M}$ be a regular class of rings containing no or all nonzero zero rings. The following statements are equivalent:*

(i) *$\mathcal{U}\mathcal{M}$ is hereditary*
(ii) *$\mathcal{M}_k\subseteqq S\mathcal{U}\mathcal{M}$*
(iii) *$\mathcal{U}\mathcal{M}=\mathcal{U}\mathcal{M}_k$*
(iv) *$\mathcal{U}\mathcal{M}$ has the intersection property relative to $\mathcal{M}_k$*
(v) *$\mathcal{U}\mathcal{M}\cap\mathcal{M}_k=0$.*

PROOF. All of the implications above, with the exception of (iii)⇒(iv), can be proved by the same methods that was used in Theorem 7 of [6] and Theorem 2 of [1]. The implication (iii)⇒(iv) follows from Theorem 6.

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RADICALS OF RINGS AND SUBRINGS

By

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1. Introduction. In this paper we examine the relationship between radicals of nonassociative rings and their subrings in certain special situations. The main result of Section 2 is based upon the notion of H-relation introduced in [5]. Suppose W is a class of nonassociative rings (perhaps, indeed, all associative) which is closed under homomorphic images and hereditary to subrings. A relation σ on W is called an H-relation if the following three properties are satisfied:

(1) $L\sigma A$ implies that L is a subring of A.
(2) If $L\sigma A$ and Φ is a homomorphism of A, then $\Phi(L)\sigma\Phi(A)$.
(3) If $L\sigma A$ and $J\lhd A$, then $L\cap J\sigma J$.

A number of examples are given in [5] and [10], among them being either an ideal, left ideal, subring, quasi-ideal or bi-ideal of the containing ring. The paper [5] for the most part gives applications of this concept to the lower radical construction. R. WIEGANDT [10] studies the relationship between semisimple classes and H-relations.

Let P be a radical. An ideal L of a ring A is called a qP-ideal (as in [8]) if $P(A/L)==0$. If σ is an H-relation, P is said to be σ-hereditary if $L\sigma A\in P$ implies $L\in P$. The main result of Section 2 then gives a relationship between qP-ideals of a ring A and qP-ideals of its σ-subrings when P is σ-hereditary. Some applications of this result are given.

For one of these applications some discussion of polynomials is needed. For a set Ω, a nonassociative ring R is called an Ω-ring if there are operations $\Omega\times R\to R$ and $R\times\Omega\to R$ which distribute over addition in R. Monomials are strings consisting of elements of Ω, a formal element 1 and nonassociating, noncommuting indeterminates x_i, with parentheses indicating the order of multiplication. A monomial must contain either some x_i or 1. Ω may be a subring of R (usually its centre or some other associative, commutative ring over which R is an algebra) or may contain special symbols denoting either involutions or automorphisms of R. The monomial h, $h\in\Omega$, is identified with $h1$ and $1h$. A polynomial is a formal sum of monomials. A polynomial f is completely homogeneous if f is homogeneous in each x_i. The height of a monomial is its degree minus the number of indeterminates occurring in it, and the height of a polynomial is the maximal height of its monomials. Given $f(x_1, \dots, x_m)$, we can linearize on x_j as follows:

$$\Delta_j f(x_1, \dots, x_{m+1}) = f(x_1, \dots, x_j+x_{m+1}, \dots, x_m)-$$

$$-f(x_1, \dots, x_j, \dots, x_m)-f(x_1, \dots, x_{m+1}, \dots, x_m).$$

In particular $\Delta_j f$ has lower height than f. For any polynomial f and ring R, let $R^+(f)$ denote the additive subgroup of R generated by all $f(r_1, \ldots, r_m)$ with $r_1, \ldots, r_m \in \in R$. We can now define the concept we need, that of an R-stable polynomial f, by induction on height as follows:

(1) All multilinear polynomials are R-stable.
(2) If f is completely homogeneous, then f is R-stable if $\Delta_j f$ is stable for every j.
(3) If f is not completely homogeneous then f is R-stable if when f is written as a sum $f_1 + \ldots + f_k$ of completely homogeneous polynomials with minimal k, then f_u is R-stable and $R^+(f_u) < R^+(f)$ for $1 < u < k$.

R-stable identities pass to central extensions and in particular the polynomial ring $R[\lambda]$. For a discussion of these concepts and their uses in the theory of polynomial identities of nonassociative rings, see Rowen [7]. These concepts are used in Section 2 to obtain sufficient conditions for the semisimplicity of the centre Z of a nonassociative ring R to imply the semisimplicity of R (relative to a radical P).

In the third section we study a property of N-radicals in associative rings. A radical P is an N-radical if it is left strong, left hereditary and supernilpotent. This notion was introduced by Sands [8] and has also been studied by Jaegermann and Sands [3] and Wiegandt [9]. Here we study conditions under which the P-radicality or P-semisimplicity passes between a ring A and subring containing a right ideal of A.

2. A theorem on σ-hereditary radicals and some applications. If P is a radical class, we denote its semisimple class by

$$SP = \{A \in W\colon A \text{ has no nonzero ideal in } P\}.$$

Theorem 1. *Let σ be an* H-*relation on W, P a σ-hereditary radical. Let $R \in W$ and suppose $R_1 \sigma R$, $B_1 \lhd R_1$ and $R_1/B_1 \in SP$. Let B be an ideal of R maximal with respect to $B \cap R_1 \subseteqq B_1$. Then $R/B \in SP$.*

Proof. If $P(R/B) = A/B \neq 0$ then $B \subset A$ and so $A \cap R_1$ is not contained in B_1 by the maximality of B. Then $A \cap R_1 \sigma A$ and so $(A \cap R_1 + B)/B \sigma A/B$ by properties (3) and (2) of H-relations. Hence $(A \cap R_1 + B)/B \in P$ since P is σ-hereditary, so that $(A \cap R_1)/(A \cap R_1 \cap B) \in P$. $(A \cap R_1)/(A \cap B)$ is a homomorphic image of this last ring, so that $(A \cap R_1 + B_1)/B_1 \cong (A \cap R_1)/(A \cap R_1 \cap B_1) = (A \cap R_1)/(A \cap B_1) \neq 0$ is a P-ideal of $R_1 B_1$. This contradiction finishes the proof.

We can now see that the condition of Theorem 1 characterizes σ-hereditary radical classes.

Theorem 2. *Let σ be an* H-*relation and P a radical. Suppose that for all $B_1 \lhd R_1 \sigma R$ such that $R_1/B_1 \in SP$ and all ideals $B \lhd R$ maximal with respect to $B \cap R_1 \subseteqq B_1$ we have $R/B \in SP$. Then P is σ-hereditary.*

Proof. Let $R \in P$, $R_1 \sigma R$, $R_1 \neq R$. Suppose $R_1 \notin P$ and so let $0 \neq R_1/B_1 \in SP$. By Zorn's lemma there is an ideal B of R maximal with respect to $B \cap R_1 \subseteqq B_1$. But then $R/B \in SP$ by hypothesis, and since $R/B \in P$ we have $B = R$, so that $B_1 = R_1$, a contradiction.

For the applications, we need the concept of a centrally admissible nonassociative ring used by Rowen in [7]. Let R be a nonassociative ring with identity and $Z = Z(R)$ its centre, the subring of R consisting of all elements of R that associate and commute with every element of R. Let λ be a commuting, associating indeterminate over Z. Then $R \otimes_Z Z[\lambda] = R[\lambda]$ is the polynomial ring over R. The nonassociative ring R is centrally admissible if it satisfies the following conditions:

(1) The intersection of the maximal ideals of $R[\lambda]$ is 0.
(2) Given a set $\{M_i : i \in I\}$ of maximal ideals of $R[\lambda]$ such that $\bigcap_{i \in I} M_i = 0$, there exists $I_1 \subseteqq I$ such that for any $I_2 \subseteqq I_1$, $R[\lambda]/\bigcap\{M_i : i \in I_2\}$ satisfies a central polynomial and $\bigcap\{M_i : i \in I\} = 0$.

Theorem 3 [7, Theorem 3.3]. *If R is centrally admissible then every nonzero ideal of R has nonzero intersection with $Z(R)$.*

Theorem 4. *Suppose P is a radical hereditary to subrings which are associative and commutative. Let R be a centrally admissible nonassociative ring with identity with centre Z and suppose $Z \in SP$. Then $R \in SP$.*

Proof. By Theorem 3 every nonzero ideal of R has nonzero intersection with Z. Let B be any ideal of R maximal with respect to $B \cap Z = 0$; thus B must be 0. Therefore $R/B = R \in SP$ by Theorem 1, which applies since σ="is a commutative, associative subring of" is an H-relation.

Clearly this result is valid in the category of Ω-rings (in [7] the definition of centrally admissible ring and Theorem 3 are stated for Ω-rings). Let B denote the Brown—McCoy radical, that is, $B(R)$ is the intersection of the maximal ideals of R.

Theorem 5 [7, Theorem 3.4]. *If a nonassociative ring R with identity is prime with R-stable central polynomial and $B(R[\lambda]) = 0$, then R is centrally admissible.*

Corollary 6. *Let P be a radical hereditary to associative, commutative subrings. If a nonassociative ring R with identity is prime with R-stable central polynomial, $B(R[\lambda]) = 0$ and $Z \in SP$, then $R \in SP$.*

Proof. This is immediate from Theorems 4 and 5.

3. Subrings containing right ideals. In this section we restrict our attention to the class of associative rings and to N-radicals in this universal class. We require three theorems proved by other authors, and we state them here for convenience.

Theorem 7 [1, Theorem 1.1]. *Let S be a ring with identity 1, A a right ideal of S such that $SA = S$ and R a subring of S containing A. Then there exists a positive integer n and an idempotent $e \in R_n$ such that $S \approx eR_n e$, where R_n denotes the $n \times n$ matrix ring over R.*

Theorem 8 [2, Theorem 1.9]. *If N is a normal radical property and $e = e^2$ is an idempotent in R, then $N(eRe) = eN(R)e = N(R) \cap eRe$.*

Theorem 9 [4, Theorem 6]. *If P is a radical class which is right or left hereditary and right or left strong, then $P(R_n) = P(R)_n$ for each ring R and each positive integer n.*

We can now prove:

Theorem 10. *Let S be an associative ring with identity 1, A a right ideal of S such that $SA = S$ and R a subring, containing 1, of S. Suppose $R \supset A$. Let P be an N-radical. Then R is P-radical iff S is P-radical, and R is P-semisimple iff S is P-semisimple.*

PROOF. Since $R\supset A$, by Theorem 7 there exists an integer $n>1$ and an idempotent $e\in R_n$ such that $S\approx eR_ne$. By Theorem 8, since every N-radical is normal, $P(S)=P(eR_ne)=eP(R_n)e=P(R_n)\cap eR_ne=P(R_n)\cap S$. By Theorem 9 this equals $P(R)_n\cap S$. Suppose $R\in P$. Then by Theorem 9, $R_n\in P$ and so $P(S)=R_n\cap P=S$.

Next suppose $S\in P$. Then $P(S)=P(R)_n\cap S=eP(R)_ne=eR_ne$. Hence $P(R)=R$ and so $R\in P$.

Suppose now that $P(S)=0$. Then $P(S)=P(R)_n\cap S=eP(R)_ne=0$. In particular, since $e\in S$, $P(R)=0$.

Finally, suppose $P(R)=0$. Then $P(S)=P(R)_n\cap S=0$. This completes the proof.

A right ideal A of S is called semimaximal if A is a finite intersection of maximal right ideals of S. If A is a semimaximal right ideal of S, then there exists a right ideal $A'\supseteqq A$ such that $SA'=S$ and $I(A')=I(A)$ where $I(A)$ is the idealizer of A in S, by [6, Proposition 1.7]. Hence we have the

COROLLARY 11. *Let S be an associative ring with identity and A a semimaximal right ideal with idealizer $I(A)$. Let P be an N-radical. Then S is P-radical iff $I(A)$ is P-radical, and S is P-semisimple iff $I(A)$ is P-semisimple.*

Clearly this corollary can be extended to the situation in which we form a sequence $S=S_0\supseteqq S_1\supseteqq S_2\supseteqq\dots$ of subrings each of which is the idealizer of a semimaximal right ideal of its predecessor, as in [6, page 60] where any ring occurring in such a sequence is called an iterated idealizer obtained from S.

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SEMISIMPLE AND TORSIONFREE CLASSES

By

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1. Preliminaries

The present note can be considered as a continuation of the paper [4]. It is the purpose of this paper to investigate the interrelations among properties satisfied by semisimple and torsionfree classes. By doing so, we shall obtain numerous characterizations of semisiple classes and torsionfree classes. Thus, for instance, semisimple classes are characterized by conditions (A^*), (b_0^*), (c) and (5) or by (A^*), (b^*) and (E), or by (B^*), (G) and (g); further, a class is a torsionfree class if and only if it satisfies conditions (b^*), (E) and (L). Dualizing some of our results also radical classes will be characterized.

In what follows we shall work in a *universal class* $\mathbf{A}$ of rings (that is, $\mathbf{A}$ is a nonempty hereditary and homomorphically closed class). A subclass $\mathbf{C}$ of $\mathbf{A}$ will be always supposed to be an *abstract class* (that is, closed under isomorphisms and containing the one-element ring 0). A *ring* will always mean a not necessarily associative ring. All results are valid also for multioperator groups (so e.g. for near-rings), but for the sake of simplicity we shall speak only of rings.

As far as associative and alternative rings are concerned, there the semisimple classes are exactly the torsionfree classes, and much stronger results can be obtained (cf. [1], [3], [6] and [7]).

Semisimple classes and torsionfree classes were treated recently in [2], [4] and [5]. For the fundamental definitions and properties of radical and semisimple classes we refer to [8], though we shall recall some definitions used in this paper.

We define the usual upper radical operator $\mathcal{U}$ and semisimple operator $\mathcal{S}$ acting on classes $\mathbf{C}$ as follows:

$\mathcal{U}\mathbf{C}=\{A\in\mathbf{A}: A$ has no nonzero homomorphic image in $\mathbf{C}\}$,

$\mathcal{S}\mathbf{C}=\{A\in\mathbf{A}: A$ has no nonzero ideal in $\mathbf{C}\}$.

As is well-known, if the class $\mathbf{C}$ is a regular class (that is, it satisfies condition (A^*)), then $\mathcal{U}\mathbf{C}$ is a radical class and if $\mathbf{C}$ is a radical class, then $\mathcal{S}\mathbf{C}$ is a semisimple class. Moreover, let us remark that the class $\mathcal{U}\mathbf{C}$ is always homomorphically closed and if $\mathbf{C}$ is homomorphically closed, then $\mathcal{S}\mathbf{C}$ is closed under subdirect sums in the following sense: if $A\in\mathbf{A}$ is a subdirect sum of $\mathcal{S}\mathbf{C}$-rings, then $A\in\mathcal{S}\mathbf{C}$.

Also, let us associate to any subclass $\mathbf{C}\subseteqq\mathbf{A}$ and to any ring $A\in\mathbf{A}$ the ideals

$$\mathbf{C}(A)=\sum_\alpha (I_\alpha\lhd A: I_\alpha\in\mathbf{C})$$

and

$$(A)\mathbf{C}=\bigcap_\alpha (J_\alpha\lhd A: A/J_\alpha\in\mathbf{C}).$$

We shall adopt the following conditions using the numbering from [4] each of them satisfied by semisimple classes.

(A*) $\mathbf{C}$ is a regular class, that is, if $A\in\mathbf{C}$, then every nonzero ideal of A has a nonzero homomorphic image in $\mathbf{C}$.

(B*) If every nonzero ideal of a ring A has a nonzero homomorphic image in the class $\mathbf{C}$, then also $A\in\mathbf{C}$.

(b*) $\mathbf{C}$ is closed under subdirect sums (in the sense as indicated above), equivalently $A/(A)\mathbf{C}\in\mathbf{C}$ for every $A\in\mathbf{A}$.

(b_0^*) $\mathbf{C}$ has the coinductive property, that is, if $I_1\supseteqq\ldots\supseteqq I_\alpha\supseteqq\ldots$ is a descending chain of ideals of a ring A such that each $A/I_\alpha\in\mathbf{C}$, then also $A/\bigcap I_\alpha\in\mathbf{C}$.

(c) $\mathbf{C}$ is closed under extensions, that is, if $I\lhd A\in\mathbf{A}$, $I\in\mathbf{C}$ and $A/I\in\mathbf{C}$, then also $A\in\mathbf{C}$.

(e*) $\big((A)\mathbf{C}\big)\mathbf{C}=(A)\mathbf{C}$ for every ring $A\in\mathbf{A}$ (or equivalently: $(A)\mathbf{C}\in\mathcal{U}\mathbf{C}$).

(2*) $\mathcal{U}\mathbf{C}(A)=(A)\mathbf{C}$ for every ring $A\in\mathbf{A}$.

(4) $\big((A)\mathbf{C}\big)\mathbf{C}\lhd A$ for every ring $A\in\mathbf{A}$.

We shall frequently refer to the following conditions satisfied by semisimple classes.

(D) If $(A)\mathbf{C}=0$, then $(A)\mathcal{S}\mathcal{U}\mathbf{C}=0$ (that is, $A\in\mathcal{S}\mathcal{U}\mathbf{C}$).

(E) If $A\in\mathcal{S}\mathcal{U}\mathbf{C}$, then $(A)\mathbf{C}=0$ $\big($that is, $(A)\mathcal{S}\mathcal{U}\mathbf{C}=0$ implies $(A)\mathbf{C}=0\big)$.

A reformulation of (B*) and (A*), respectively, is

(F) If $(A)\mathcal{S}\mathcal{U}\mathbf{C}=0$, then $A\in\mathbf{C}$.

(G) If $A\in\mathbf{C}$, then $(A)\mathcal{S}\mathcal{U}\mathbf{C}=0$.

Further properties of semisimple classes are

(e_1^*) $K\in\mathcal{U}\mathbf{C}$ for any ring A and for any ideal K of A which is minimal with respect to $A/K\in\mathbf{C}$.

(f) If $A/B\in\mathbf{C}$ and $A/C\in\mathbf{C}$, then also $A/(B\cap C)\in\mathbf{C}$.

(5) If $B\lhd A$ and B is minimal with respect to $A/B\in\mathbf{C}$ and $C\lhd B$ and C is minimal with respect to $B/C\in\mathbf{C}$, then $C\lhd A$.

2. Semisimple classes

Let us recall that a subclass $\mathbf{S}$ of $\mathbf{A}$ is called a *semisimple class* if and only if $\mathbf{S}$ satisfies conditions (A*) and (B*).

THEOREM 1. *For a subclass* $\mathbf{S}$ *of* $\mathbf{A}$ *the following conditions are equivalent:*

I. $\mathbf{S}$ *is a semisimple class.*
II. $\mathbf{S}$ *satisfies conditions* (A*), (b*), (c) *and* (4).
III. $\mathbf{S}$ *satisfies conditions* (A*), (b*) *and* (e*).
IV. $\mathbf{S}$ *satisfies conditions* (A*), (b_0^*) *and* (e_1^*).

PROOF. I$\Leftrightarrow$II is the assertion of [4] Theorem 10 and I$\Leftrightarrow$III is Mlitz' Theorem 4.2 in [5].

IV$\Rightarrow$III. Using (b_0^*) and Zorn's lemma, it is clear that every ring A has an ideal K which is minimal with respect to the property $A/K\in\mathbf{S}$. Moreover, by (e_1^*) also $K\in\mathcal{U}\mathbf{S}$ holds. Let I be any ideal of A such that $A/I\in\mathbf{S}$. Then

$$K/(K\cap I)\cong(K+I)/I\lhd A/I\in\mathbf{S}$$

holds. Applying (A*) it follows that $K/(K\cap I)$ has a nonzero homomorphic image B in $\mathbf{S}$, provided that $K/(K\cap I)\neq0$. This is, however, impossible, for B is also a homomorphic image of $K\in\mathcal{U}\mathbf{S}$. Thus $K=K\cap I$ holds implying $K\simeq I$. Hence K is the unique ideal which is minimal with respect to $A/K\in\mathbf{S}$ and consequently $K=(A)\mathbf{S}$ holds. $K=(A)\mathbf{S}$ implies immediately the validity of (b*) and (e*).

II$\Rightarrow$IV. The validity of (b_0^*) follows from (b*). Using (b*) it follows that $K=(A)\mathbf{S}$ is the unique ideal which is minimal with respect to the property $A/K\in\mathbf{S}$. Since the class $\mathbf{S}$ is a semisimple class (cf. [4] Theorem 10), in view of [4] Proposition 7 or [5] Theorem 1 the class $\mathbf{S}$ satisfies condition (2*). Thus $K=(A)\mathbf{S}=\mathcal{U}\mathbf{S}(A)\in\mathcal{U}\mathbf{S}$ holds establishing (e_1^*). Q.e.d.

Next, we are going to prove interrelations among conditions occurring in characterizations of semisimple classes.

PROPOSITION 1. *A class* $\mathbf{C}$ *satisfies condition* (b*) *if and only if* $\mathbf{C}$ *satisfies* (b_0^*) *and* (f).

PROOF. The necessity is trivial.

Assume that the class $\mathbf{C}$ satisfies (b_0^*) and (f) and let us consider a subdirect sum $A=\sum\limits_{\alpha\in\Lambda}(A_\alpha : A_\alpha\in\mathbf{C})$ belonging to $\mathbf{A}$. Then there exist ideals I_α, $\alpha\in\Lambda$, of A such that $A_\alpha\cong A/I_\alpha\in\mathbf{C}$ and $\bigcap\limits_{\alpha\in\Lambda}I_\alpha=0$. Using ($b_0^*$) and Zorn's lemma, A contains an ideal K such that $A/K\in\mathbf{C}$ and K is minimal with respect to this property. Let us consider an ideal L of A such that $A/L\in\mathbf{C}$. Applying (f) we get $A/(K\cap L)\in\mathbf{C}$. Hence the choice of K implies $K\subseteqq L$, and so also $K\subseteqq\bigcap\limits_{\alpha\in\Lambda}I_\alpha=0$ holds. Thus $A\in\mathbf{C}$ follows and (b*) is fulfilled. Q.e.d.

PROPOSITION 2. *In a class* $\mathbf{C}$ *the validity of* (5) *and* (b*) *imply that of* (4).

PROOF. By (b*) the ideal $B=(A)\mathbf{C}$ is the unique minimal ideal of A such that $A/B\in\mathbf{C}$. By the same reason $C=(B)\mathbf{C}$ is the unique minimal ideal of B such that $B/C\in\mathbf{C}$. Hence (5) yields

$$\big((A)\mathbf{C}\big)\mathbf{C}=(B)\mathbf{C}=C\lhd A$$

which means the validity of (4). Q.e.d.

PROPOSITION 3. (b*) *and* (4) *imply* (5).

PROOF. By (b*) the ideal $B=(A)\mathbf{C}$ is the unique one which is minimal with respect to $A/B\in\mathbf{C}$, and the corresponding assertion applies for $C=(B)\mathbf{C}\lhd B$. Hence using (4) we get

$$C=(B)\,\mathbf{C}=\big((A)\,\mathbf{C}\big)\,\mathbf{C}\lhd A$$

proving the validity of (5). Q.e.d.

PROPOSITION 4. *If a class* $\mathbf{C}$ *satisfies conditions* (A*), (b_0^*), (c) *and* (5), *then also* (b*) *is satisfied.*

PROOF. Using Zorn's lemma and (b_0^*) in any ring A there exists an ideal B which is minimal with respect to $A/B\in\mathbf{C}$. Similarly there is an ideal C of B being minimal with respect to $B/C\in\mathbf{C}$. By (5) we have $C\lhd A$ and so the isomorphism

$$\frac{A/C}{B/C}\cong A/B$$

and (c) yields that $A/C\in\mathbf{C}$. Hence $B\subseteqq C$ follows. Thus $B=C$ and B has no nonzero homomorphic image in $\mathbf{C}$ and so $(B)\mathbf{C}=B$ is valid.

Let D denote any ideal of A such that $A/D\in\mathbf{C}$. Since

$$B/(B\cap D)\cong(B+D)/D\lhd A/D\in\mathbf{C},$$

condition (A*) implies that $B/(B\cap D)$ has a nonzero homomorphic image K in $\mathbf{C}$, provided that $B/(B\cap D)\neq 0$. K is, however, also a homomorphic image of B, hence $K=0$ implying $B/(B\cap D)=0$. Thus $B\subseteqq D$ and also $B\subseteqq(A)\mathbf{C}\subseteqq B$ hold. Hence it follows $A/(A)\mathbf{C}=A/B\in\mathbf{C}$ proving the validity of (b*). Q.e.d.

THEOREM 2. *The following conditions are characterizing the semisimple classes:*

V. $\mathbf{S}$ *satisfies* (A*), (b_0^*), (c), (f) *and* (4).

VI. $\mathbf{S}$ *satisfies* (A*), (b_0^*), (c) *and* (5).

PROOF. The equivalence V⇔I follows from Proposition 1. II⇒VI is a consequence of Propositions 1 and 3, meanwhile VI⇒II follows from Propositions 4 and 2. Q.e.d.

PROPOSITION 5. *For any subclass* $\mathbf{C}$ *of* $\mathbf{A}$, *any* $A\in\mathbf{A}$ *and any* $I\lhd A$ *the inclusion*

$$(A/I)\mathbf{C}\supseteqq((A)\mathbf{C}+I)/I$$

holds. Moreover, if $I\subseteqq(A)\mathbf{C}$, *then equality holds, that is* $(A)\mathbf{C}/I=(A/I)\mathbf{C}$.

PROOF. Let L_α, $\alpha\in\Lambda$, be the ideals of A such that $A/L_\alpha\in\mathbf{C}$. We have

$$((A)\mathbf{C}+I)/I=\Big(\big(\bigcap_{\alpha\in\Lambda}L_\alpha\big)+I\Big)/I\subseteqq\bigcap_{\alpha\in\Lambda}(L_\alpha+I)/I=$$

$$=\bigcap_{\alpha\in\Lambda}((L_\alpha+I)/I)\subseteqq\bigcap_{\beta\in\Lambda}(L_\beta/I\colon I\subseteqq L_\beta)=(A/I)\mathbf{C},$$

since $(A/I)/(L/I)\in\mathbf{C}$ holds if and only if $A/L\in\mathbf{C}$ $(I\subseteqq L)$. If now $I\subseteqq(A)\mathbf{C}$, that is, $I\subseteqq L$ holds for all $\alpha\in\Lambda$ then in the above proof instead of inclusion obviously equality holds and we get $(A)\mathbf{C}/I=(A/I)\mathbf{C}$.

COROLLARY. *For any two subclasses* $\mathbf{C}$, $\mathbf{D}$ *of* $\mathbf{A}$ *the inclusion*

$$(A/(A)\mathbf{D})\mathbf{C}\supseteqq((A)\mathbf{C}+(A)\mathbf{D})/(A)\mathbf{D}$$

holds. Moreover, if $\mathbf{C}\subseteqq\mathbf{D}$, then

$$(A/(A)\mathbf{D})\mathbf{C}=(A)\mathbf{C}/(A)\mathbf{D}$$

and, in particular, $(A/(A)\mathbf{C})\mathbf{C}=0$ *hold.*

PROPOSITION 6. *For any class* $\mathbf{C}$ *condition* (D) *is equivalent to either of the following two conditions:*

(i) $A/(A)\mathbf{C}\in\mathcal{S}\mathcal{U}\mathbf{C}$ *for every ring* $A\in\mathbf{A}$.

(ii) $\mathcal{U}\mathbf{C}(A)\subseteqq(A)\mathbf{C}$ *for every ring* $A\in\mathbf{A}$.

PROOF. Suppose that $\mathbf{C}$ satisfies (D). Since by the Corollary of Proposition 5 we have $(A/(A)\mathbf{C})\mathbf{C}=0$, an application of (D) yields $A/(A)\mathbf{C}\in\mathcal{S}\mathcal{U}\mathbf{C}$. Thus (i) holds.

Assume the validity of (i). Then $A/(A)\mathbf{C}$ has no nonzero ideal in $\mathcal{U}\mathbf{C}$. Let $K\neq 0$ be an ideal of A such that $K\in\mathcal{U}\mathbf{C}$. Since the class $\mathcal{U}\mathbf{C}$ is homomorphically closed, it follows

$$(K+(A)\mathbf{C})/(A)\mathbf{C}\cong K/(K\cap(A)\mathbf{C})\in\mathcal{U}\mathbf{C}.$$

Hence $(K+(A)\mathbf{C})/(A)\mathbf{C}$ is a $\mathcal{U}\mathbf{C}$-ideal of $A/(A)\mathbf{C}$ which has to be 0. Consequently $K\subseteqq(A)\mathbf{C}$ holds and so $\mathcal{U}\mathbf{C}(A)\subseteqq(A)\mathbf{C}$ is established.

Assume that $\mathbf{C}$ satisfies (ii) and that $(A)\mathbf{C}=0$. Then (ii) implies $\mathcal{U}\mathbf{C}(A)=0$ and so A has no nonzero ideal in $\mathcal{U}\mathbf{C}$ which means $A\in\mathcal{S}\mathcal{U}\mathbf{C}$.

PROPOSITION 7. *Condition* (D) *is equivalent to* (A*).

PROOF. Let us consider a class $\mathbf{C}$ satisfying (D) and a ring $A\in\mathbf{C}$. Now $(A)\mathbf{C}=0$ holds, and so (D) implies $A\in\mathcal{S}\mathcal{U}\mathbf{C}$. Hence A has no nonzero ideal in $\mathcal{U}\mathbf{C}$, that is every nonzero ideal of A has a nonzero homomorphic image in $\mathbf{C}$.

Assume (A*) and consider a ring A such that $(A)\mathbf{C}=0$. Then $\mathbf{C}\subseteqq\mathcal{S}\mathcal{U}\mathbf{C}$ holds and so we get

$$\mathcal{U}\mathbf{C}(A)\subseteqq(A)\mathcal{S}\mathcal{U}\mathbf{C}\subseteqq(A)\mathbf{C}=0.$$

Hence $A\in\mathcal{S}\mathcal{U}\mathbf{C}$ holds proving (D). Q.e.d.

PROPOSITON 8. *Condition* (2*) *implies the validity of* (A*), (e*) *and* (4) *in any class* $\mathbf{C}$.

PROOF. Let A be any ring such that $(A)\mathbf{C}=0$. Now by (2*) we get $\mathcal{U}\mathbf{C}(A)=$ $=(A)\mathbf{C}=0$ and so A has no nonzero $\mathcal{U}\mathbf{C}$-ideal. Thus $A\in\mathcal{S}\mathcal{U}\mathbf{C}$ holds and (D) is satisfied. Hence by Proposition 7 also (A*) is fulfilled.

Since $\mathbf{C}$ satisfies (A*) the class $\mathcal{U}\mathbf{C}$ is a radical class. Again by (2*) we have $(A)\mathbf{C}=\mathcal{U}\mathbf{C}(A)\in\mathcal{U}\mathbf{C}$ for every ring A, and so (e*) holds.

(2*)⇒(4) follows from (e*)⇒(4). Q.e.d.

PROPOSITION 9. *A subclass* $\mathbf{C}$ *satisfies* (2*) *if and only if* $\mathbf{C}$ *satisfies* (A*) *and* (e*).

PROOF. The necessity follows from Proposition 8.

Conversely, assume (A*) and (e*). In view of Propositions 6 (ii) and 7 we have $\mathcal{U}\mathbf{C}(A)\subseteqq(A)\mathbf{C}$ for every ring A. From (e*) we get that $(A)\mathbf{C}\in\mathcal{U}\mathbf{C}$. Hence also $(A)\mathbf{C}\subseteqq$ $\subseteqq\mathcal{U}\mathbf{C}(A)$ holds. Q.e.d.

THEOREM 3. *A subclass* $\mathbf{S}$ *of* $\mathbf{A}$ *is a semisimple class if and only if one of the following conditions is fulfilled:*

VII. $\mathbf{S}$ *satisfies* (2*) *and* (b*).

VIII. $\mathbf{S}$ *satisfies* (D), (b*) *and* (e*).

IX. $\mathbf{S}$ *satisfies* (D), (b*), (c) *and* (4).

PROOF. The equivalence I⇔VII was shown in [4] Theorem 11. The other assertions are immediate consequences of Theorem 1, II and III and of Proposition 7. Q.e.d.

PROPOSITION 10. *In any class* $\mathbf{C}$ *condition* (2*) *or* (e) *implies*

(i) $(A)\mathbf{C}\subseteqq\mathcal{U}\mathbf{C}(A)$,

and hence also the validity of (E) *follows.*

Proof. The implication $(2^*)\Leftrightarrow$(i) is trivial. The validity of (e^*) means $(A)\mathbf{C}\in\mathcal{U}\mathbf{C}$. Hence (i) holds.

In particular, if $A\in\mathcal{S}\mathcal{U}\mathbf{C}$, then $\mathcal{U}\mathbf{C}(A)=0$, so $(A)\mathbf{C}=0$ holds and (E) is established. Q.e.d.

Proposition 11. *In any class* $\mathbf{C}$ *the following equivalence holds:* $(2^*)\Leftrightarrow(\mathrm{A}^*)$ *and* (E).

Proof. The necessity follows from Propositions 8 and 10.

Assume (A^*) and (E). Since by (A^*) the class $\mathcal{U}\mathbf{C}$ is a radical class and hence by (c) the ring $A/\mathcal{U}\mathbf{C}(A)$ has no nonzero ideal in $\mathcal{U}\mathbf{C}$, that is, $A/\mathcal{U}\mathbf{C}(A)\in\mathcal{S}\mathcal{U}\mathbf{C}$. Applying (E) we get $(A/\mathcal{U}\mathbf{C}(A))\mathbf{C}=0$, and so in view of the Corollary of Proposition 5 we have $(A)\mathbf{C}\leqq\mathcal{U}\mathbf{C}(A)$. By Propositions 6 and 7 (A^*) is equivalent to $\mathcal{U}\mathbf{C}(A)\subseteqq(A)\mathbf{C}$ for all $A\in\mathbf{A}$. Hence $(A)\mathbf{C}=\mathcal{U}\mathbf{C}(A)$ is valid proving (2^*). Q.e.d.

Theorem 3. VII and Proposition 11 yield immediately

Theorem 4. *A subclass* $\mathbf{S}$ *of* $\mathbf{A}$ *is a semisimple class if and only if*
X. $\mathbf{S}$ *satisfies* (A^*), (b^*) *and* (E).

Characterization X sharpens Mlitz' Theorem 4.2 in [5] inasmuch as (e^*) implies (E) by Proposition 10.

Proposition 12. *Condition* (F) *is equivalent to*
(i) $A/(A)\mathcal{S}\mathcal{U}\mathbf{C}\in\mathbf{C}$ *for every ring* $A\in\mathbf{A}$.

Proof. (F)$\Leftrightarrow$(i) is trivial in view of the Corollary of Proposition 5.

Proposition 13. *Condition* (E) *is equivalent to*
(i) $(A)\mathbf{C}\subseteqq(A)\mathcal{S}\mathcal{U}\mathbf{C}$ *for every ring* $A\in\mathbf{A}$,
and condition (D) *is equivalent to*
(ii) $(A)\mathcal{S}\mathcal{U}\mathbf{C}\subseteqq(A)\mathbf{C}$ *for every ring* $A\in\mathbf{A}$.

Proof. Assume the validity of (E). By the Corollary of Proposition 5 we have $(A/(A)\mathcal{S}\mathcal{U}\mathbf{C})\mathcal{S}\mathcal{U}\mathbf{C}=0$ and so (E) yields $(A/(A)\mathcal{S}\mathcal{U}\mathbf{C})\mathbf{C}=0$, and hence the Corollary of Proposition 5 implies $(A)\mathbf{C}\subseteqq(A)\mathcal{S}\mathcal{U}\mathbf{C}$.

(i)$\Rightarrow$(E) is trivial.

The proof of the second equivalence is similar. Q.e.d.

Proposition 14. *The following implication holds:* $(\mathrm{B}^*)\Rightarrow$(E).

Proof. Trivially (B^*) means that $\mathcal{S}\mathcal{U}\mathbf{C}\subseteqq\mathbf{C}$ and by Proposition 13 (i) (B^*) implies (E).

Let us remark that (E) does not imply (B^*), since $(A)\mathbf{C}=0$ does not imply $A\in\mathbf{C}$ in general.

Theorem 5. *A further characterization of a semisimple class* $\mathbf{S}$ *is the following:*
XI. $\mathbf{S}$ *satisfies* (D) *and* (F).

The statement follows from Proposition 7. Q.e.d.

3. Torsionfree classes

In this section we are going to characterize torsionfree classes by using the results of section 2. In order to deal with torsionfree classes, we introduce the following conditions a class may satisfy.

(a^*) $\mathbf{C}$ is hereditary, that is, $I\lhd A\in\mathbf{C}$ implies $I\in\mathbf{C}$.
(1^*) $(I)\mathbf{C}\subseteqq(A)\mathbf{C}$ for every ideal I of any ring $A\in\mathbf{A}$.
(L) If $(A)\mathbf{C}=0$, then $(I)\mathbf{C}=0$ for every ideal I of any $A\in\mathbf{A}$.

Proposition 15. *Conditions* (a^*) *and* (b^*) imply (L).

Proposition 16. *Condition* (1^*) *implies*
(i) $(I)\mathbf{C}\subseteqq I\cap(A)\mathbf{C}$
for any ideal I *of any ring* $A\in\mathbf{A}$, *moreover,* (i) *implies* (L).

Proposition 17. *Condition* (L) *implies* (A^*).

Proof. If $I\lhd A\in\mathbf{C}$ and I has no nonzero homomorphic image in $\mathbf{C}$, then by definition $(I)\mathbf{C}=I$. But $A\in\mathbf{C}$ gives $(A)\mathbf{C}=0$, so (L) implies $I=(I)\mathbf{C}=0$. Hence every nonzero ideal of A has a nonzero homomorphic image in $\mathbf{C}$ and (A^*) holds. Q.e.d.

Proposition 18. *Conditions* (b^*) *and* (L) *imply* (a^*). *Thus* (a^*) *and* (b^*) *are equivalent to* (b^*) *and* (L).

Proof. Let I be an ideal of a ring $A\in\mathbf{C}$. Now $(A)\mathbf{C}=0$ holds and by (L) also $(I)\mathbf{C}=0$ is valid. Hence by (b^*) it follows $I\in\mathbf{C}$. The second statement follows from Proposition 15. Q.e.d.

Proposition 19. *Conditions* (b^*) *and* (L) *imply* (1^*). *Thus for any subclass* $\mathbf{C}$ *satisfying* (b^*), *conditions* (1^*) *and* (L) *are equivalent.*

Proof. By (b^*) we have $A/(A)\mathbf{C}\in\mathbf{C}$. Hence in view of the Corollary of Proposition 5 we have $(A/(A)\mathbf{C})\mathbf{C}=0$ and so (L) and (b^*) imply

$$I/(I\cap(A)\mathbf{C})\cong(I+(A)\mathbf{C})/(A)\mathbf{C}\in\mathbf{C}$$

for any ideal I of A. Hence we get $(I)\mathbf{C}\subseteqq I\cap(A)\mathbf{C}$ that is, $(I)\mathbf{C}\subseteqq(A)\mathbf{C}$.

The second assertion follows from Proposition 16. Q.e.d.

A hereditary semisimple class is called a *torsionfree class* (cf. [2] and [4]). Hence *a class* $\mathbf{F}$ *is a torsionfree class if and only if* $\mathbf{F}$ *satisfies conditions* (a^*) *and* (B^*).

In view of Propositions 7, 12, 17, 18 and 19 we get immediately the following

Theorem 6. *Substituting condition* (A^*) *and* (D), *respectively, by either* (a^*), *or* (1^*) *or* (L) *in the eight characterizations* I—VI *and* X—XI, *one gets characterizations of torsionfree classes.*

Let us remark that by Propositions 7, 9 and 12 the addition of (a^*) (or (1^*) or (L), respectively), to VII—IX will not yield other characterization of torsionfree classes than those listed in Theorem 6.

4. Remarks concerning radical classes

In the context of characterizing semisimple classes, it is worth mentioning that the dual conditions of III, IV and X give new characterizations of radical classes by relatively weak conditions.

First of all, let us observe that the categorically dual condition to (e*) is condition

(e) $\mathbf{R}(A/\mathbf{R}(A))=0$ for every $A\in\mathbf{A}$ (or equivalently: $A/\mathbf{R}(A)\in\mathcal{S}\mathbf{R}$).

Usually *radical classes* are defined as classes $\mathbf{R}$ satisfying the following three conditions: (e) and

(a) $\mathbf{R}$ is homomorphically closed,

(b) $\mathbf{R}(A)\in\mathbf{R}$.

In characterizing radical classes we shall make use of the following conditions.

(A) If $A\in\mathbf{R}$, then every nonzero homomorphic image of A has a nonzero ideal in $\mathbf{R}$.

(b_0) $\mathbf{R}$ has the inductive property, that is, if $I_1\subseteqq\ldots\subseteqq I_\alpha\subseteqq\ldots$ is an ascending chain of $\mathbf{R}$-ideals of a ring A, then also $\bigcup I_\alpha\in\mathbf{R}$ holds.

(e_1) $A/K\in\mathcal{S}\mathbf{R}$ holds for any ring $A\in\mathbf{A}$ and for any maximal $\mathbf{R}$-ideal K of A.

Obviously, conditions (A), (a), (b), (b_0) and (e_1) are dual to those written with asterisk.

THEOREM 7. *The following four conditions are equivalent for a subclass* $\mathbf{R}$ *of* $\mathbf{A}$;

I. $\mathbf{R}$ *is a radical class.*

II. $\mathbf{R}$ *satisfies conditions* (A), (b) *and* (e).

III. $\mathbf{R}$ *satisfies conditions* (A), (b_0) *and* (e_1).

IV. $\mathbf{R}$ *satisfies conditions* (a), (b_0) *and* (e_1).

PROOF. The implications I⇒II and II⇒III are trivial.

III⇒IV. Assume that the class $\mathbf{R}$ is not homomorphically closed. Then there is a ring $A\in\mathbf{R}$ having a factor ring $A/I\notin\mathbf{R}$. By (A) the ring A/I has a nonzero ideal in $\mathbf{R}$. Hence by (b_0) and by Zorn's lemma there exists a maximal $\mathbf{R}$-ideal K/I of A/I. Using (e_1) we get

$$0\neq A/K\cong\frac{A/I}{K/I}\in\mathcal{S}\mathbf{R}.$$

On the other hand by (A) the ring A/K, as a homomorphic image of A, has a nonzero ideal in $\mathbf{R}$ and consequently $A/K\notin\mathcal{S}\mathbf{R}$ holds. This contradiction proves that $\mathbf{R}$ is a homomorphically closed.

IV⇒I: By (b_0) and by Zorn's lemma any ring A has a maximal $\mathbf{R}$-ideal K. Let I be any $\mathbf{R}$-ideal of A. Using (a) we have $(K+I)/K\cong I/(K\cap I)\in\mathbf{R}$. Since by ($e_1$) $(K+I)/K\triangleleft\triangleleft A/K\in\mathcal{S}\mathbf{R}$ holds, it follows $(K+I)/K=0$. Hence $I\subseteqq K$ for every $\mathbf{R}$-ideal I of A. Consequently $\mathbf{R}(A)=K$ is valid implying the validity of (b) and (e). Q.e.d.

THEOREM 8. *A subclass* $\mathbf{R}$ *of* $\mathbf{A}$ *is a radical class if and only if* $\mathbf{R}$ *satisfies conditions* (A), (b) *and*

($\bar{\mathrm{E}}$) *if* $A\in\mathcal{U}\mathcal{S}\mathbf{R}$, *then* $\mathbf{R}(A)=A$.

PROOF. Evidently the above conditions are necessary.

For the sufficiency let us remind that (A) implies $\mathbf{R}\subseteqq\mathcal{U}\mathcal{S}\mathbf{R}$ (cf. [4] Proposition 1). Furthermore, ($\bar{\mathrm{E}}$) and (b) yield readily $\mathcal{U}\mathcal{S}\mathbf{R}\subseteqq\mathbf{R}$. Hence $\mathbf{R}=\mathcal{U}\mathcal{S}\mathbf{R}$ holds.

We claim that the class $\mathcal{S}\mathbf{R}$ satisfies (A*). Let $I\neq 0$ be an ideal of a ring $A\in\mathcal{S}\mathbf{R}$, If I has no nonzero homomorphic image in $\mathcal{S}\mathbf{R}$, then we have $I\in\mathcal{U}\mathcal{S}\mathbf{R}=\mathbf{R}$. This is, however, impossible, so (A*) holds. Thus the class $\mathbf{R}=\mathcal{U}\mathcal{S}\mathbf{R}$ is a radical class.

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УСТОЙЧИВОСТЬ КВАДРАТУРНЫХ ФОРМУЛ ГАУССА—ЯКОБИ И НЕУСТОЙЧИВОСТЬ СХОДИМОСТИ В СРЕДНЕМ ИНТЕРПОЛЯЦИОННОГО ПРОЦЕССА ЛАГРАНЖА ПРИ РАСШИРЕНИИ МАТРИЦЫ УЗЛОВ ИНТЕРПОЛИРОВАНИЯ

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1. На практике при решении различных задач связанных с интерполяцией или с механическими квадратурами иногда приходится к известным узлам добавлять в качестве узлов заданные фиксированные точки. На первый взгляд может показаться, что от такого расширения матрицы узлов улучшается сходимость процесса. Однако это может быть и не так. Для простоты рассмотрим систему узлов

$$x_0^{(n+2)} = -1,\ x_k^{(n+2)} = \cos\frac{2(n-k)+1}{2n}\pi, \qquad k = 1, \ldots, n,\ x_{n+1}^{(n+2)} = 1,\ n = 1, 2, \ldots, \tag{1}$$

которая получается из системы узлов П. Л. Чебышева

$$x_k^{(n)} = \cos\frac{2k-1}{2n}\pi, \quad k = 1, \ldots, n, \quad n = 1, 2, \ldots \tag{2}$$

добавлением в качестве узлов точек $x=\pm 1$. В [1] было доказано, что интерполяционный процесс Эрмита—Фейера, построенный при матрице узлов (1) для функции $f(x)=|x|$, расходится при $x=0$. Этот результат неожиданный, ибо согласно классическому результату Л. Фейера [2] процесс Эрмита—Фейера, построенный для любой непрерывной в $[-1, 1]$ функции при узлах Чебышева (2), равномерно сходится к этой функции в $[-1, 1]$. Позже было доказано [3], что при $f(x)=x^2$ и узлах (1) процесс Эрмита—Фейера расходится всюду в $(-1, 1)$. В силу указанных отрицательных результатов возникают следующие вопросы.

1) Что происходит с квадратурными формулами типа Гаусса—Якоби при расширении матрицы узлов интерполирования и сохранении веса?

2) Сохраняется ли известная теорема Эрдёша—Турана—Шохата [4], [5] при расширении матрицы узлов интерполирования и сохранении веса?

3) Сходится ли интерполяционно-квадратурный процесс, построенный для любой функции непрерывной в $[-1, 1]$ при расширенной системе узлов (1)? В настоящей работе даются ответы на поставленные вопросы.

2. Пусть заданы точки $x_1^{(n)} < x_2^{(n)} < \ldots < x_n^{(n)}$. Через $\{l_k^{(n)}(x)\}_{k=1}^n$ обозначим фундаментальные полиномы Лагранжа степени $(n-1)$, построенные для этих точек. Будем говорить, что квадратурная формула [6]

$$\int_{-1}^{1} f(x)\,p(x)\,dx \approx \sum_{k=1}^{n} A_k^{(n)} f(x_k^{(n)}), \quad A_k^{(n)} = \int_{-1}^{1} l_k^{(n)}(x)\,p(x)\,dx, \tag{3}$$

типа Гаусса—Якоби, если числа $\{x_k^{(n)}\}_{k=1}^n$ являются корнями ортогонального полинома степени n веса $p(x)$.

Теорема 1. *Пусть* $\{x_k\}_{k=1}^n$ *— корни ортогонального полинома веса* $p(x)$ *и* $\{a_i\}_{i=1}^n$ *— произвольные точки из* $[-1, 1]$, *отличные от точек* $\{x_k\}_{k=1}^n$. *Тогда квадратурная формула Гаусса—Якоби по узлам* $\{x_k\}_{k=1}^n$ *с весом* $p(x)$ *совпадает с квадратурной формулой по узлам* $\mathfrak{M}=\{x_k\}_{k=1}^n\cup\{a_i\}_{i=1}^n$ *с тем же весом* $p(x)$.

Это свойство квадратурных формул Гаусса—Якоби естественно называть свойством устойчивости при расширении системы узлов. Для доказательства теоремы 1 нужна следующая

Теорема 2. *Для того чтобы формула* (3) *была формулой типа Гаусса—Якоби необходимо и достаточно, чтобы выполнялись равенства*

$$\int_{-1}^{1} x^i l_k(x)\,p(x)\,dx = x_k^i \int_{-1}^{1} l_k(x)\,p(x)\,dx, \quad 0\leqq i\leqq n,\ 1\leqq k\leqq n.$$

Доказательство теоремы 1. Фундаментальные полиномы Лагранжа $\tilde l_s(x)$ расширенной системы узлов $\mathfrak{M}$ задаются одной из следующих двух формул:

$$\tilde l_s(x) = \frac{\omega_n^{(1)}(x)}{\omega_n^{(1)}(x_s)}\,l_s(x), \quad \omega_n^{(1)}(x) = \prod_{i=1}^{n}(x-a_i), \tag{4}$$

или

$$\tilde l_s(x) = \frac{\omega_n(x)}{\omega_n(a_l)}\,L_s(x), \quad \omega_n(x) = \prod_{i=1}^{n}(x-x_i),$$

где $\{l_s(x)\}$ — фундаментальные полиномы Лагранжа степени $n-1$, соответствующие узлам $\{x_l\}_{l=1}^n$, а $L_s(x)$ — фундаментальные полиномы Лагранжа степени $n-1$, соответствующие узлам $\{a_l\}_{l=1}^n$. Поэтому коэффициенты $\tilde A_s$ расширенной квадратурной формулы соответсвующие узлам $\{a_l\}_{l=1}^n$ равны нулю, ибо

$$\tilde A_s = \frac{1}{\omega_n(a_l)}\int_{-1}^{1}\omega_n(x)L_s(x)\,p(x)\,dx = 0,$$

поскольку $L_s(x)$ — многочлен степени $n-1$. Из (4) следует, что коэффициент $\tilde A_l$ расширенной квадратурной формулы, соответствующий узлу x_l равен

$$\tilde A_l = \int_{-1}^{1}\frac{\omega_n^{(1)}(x)}{\omega_n^{(1)}(x_l)}\,l_i(x)\,p(x)\,dx.$$

Поэтому согласно теореме 2 $\tilde A_l = A_i$, где A_i — i-тый коэффициент квадратурной формулы (3) по узлам $\{x_l\}_{l=1}^n$. Итак, квадратурная формула (3) по узлам $\{x_l\}_{l=1}^n$ совпадает с квадратурной формулой (3) по узлам $\mathfrak{M}$.

3. Теорема 3. *Пусть* $\{\tilde L_n(f, x)\}$ *— интерполяционный процесс Лагранжа, построенный при расширенной системе узлов* (1) *Чебышева. Существуют такие непрерывные функции для которых*

$$\varlimsup_{n\to\infty}\int_{-1}^{1}[f(x)-\tilde L_n(f, x)]^2\frac{dx}{\sqrt{1-x^2}} = \infty.$$

Эту теорему следует сравнить с теоремой Эрдёша—Турана—Шохата [4], [5] согласно которой для любой $f\in C_{[-1,1]}$

$$\lim_{n\to\infty}\int_{-1}^{1}p(x)[f(x)-L_n(f, x)]^2\,dx = 0.$$

Здесь $L_n(f, x)$ — интерполяционный многочлен Лагранжа степени $n-1$, построенный по корням ортогонального полинома степени n веса $p(x)$.

Доказательство теоремы 3. Положим $\|f-g\|=\left(\int_{-1}^{1}(f-g)^2p\,dx\right)^{1/2}$. Тогда

$$\|\tilde L_n(f)-L_n(f)\|-\|f-L_n(f)\| \leqq \|f-\tilde L_n(f)\| \leqq \|\tilde L_n(f)-L_n(f)\|+\|f-L_n(f)\|, \tag{5}$$

где $L_n(f)$ — полином Лагранжа степени $n-1$, построенный при узлах (2). Согласно теореме Эрдёша—Турана—Шохата $\|f-L_n(f)\|=o(1)$, $n\to\infty$. Поэтому неравенства (5) принимают вид:

$$\|\tilde L_n(f)-L_n(f)\|-o(1) \leqq \|f-\tilde L_n(f)\| \leqq \|\tilde L_n(f)-L_n(f)\|+o(1). \tag{6}$$

Ради простоты будем считать, что n — четное число и $f(x)$ — нечетная функция. Тогда легко видеть, что

$$\tilde L_n(f, x)-L_n(f, x) = xT_n(x)\big(f(1)-L_n(f, 1)\big), \quad T_n(x) = \cos n \arccos x. \tag{7}$$

Так как

$$\int_{-1}^{1}\frac{x^2T_n^2(x)}{\sqrt{1-x^2}}\,dx = \frac{\pi}{4},$$

то из (6) и (7) следует, что

$$|f(1)-L_n(f, 1)|\sqrt{\frac{\pi}{4}}-o(1) \leqq \|f-\tilde L_n(f)\| \leqq |f(1)-L_n(f, 1)|\sqrt{\frac{\pi}{4}}+o(1). \tag{8}$$

Известно [7], что при узлах (2) $\sum_{k=1}^{n}|l_k^{(n)}(1)|\approx\frac{2}{\pi}\ln n$. Поэтому согласно теореме Банаха—Штейнхауза существуют такие нечетные непрерывные функции для которых $\varlimsup_{n\to\infty}|f(1)-L_n(f, 1)|=\infty$. Отсюда и из (8) следует теорема 3.

4. Будем процесс механических квадратур

$$\int_{-1}^{1}f(x)\,dx \approx \sum_{k=1}^{n}A_k^{(n)}f(x_k^{(n)})$$

называть интерполяционно-квадратурным, если $A_k^{(n)}=\int_{-1}^{1}l_k^{(n)}(x)\,dx$, где $\{l_k^{(n)}(x)\}$ — фундаментальные полиномы Лагранжа точек $\{x_k^{(n)}\}$.

Теорема 4. *При расширенной системе узлов Чебышева* (1) *все коэффициенты* $A_k^{(n)}$ *интерполяционно-квадратурного процесса, за исключением первого и последнего, положительные.* $A_0^{(n)}$ *и* $A_{n+1}^{(n)}$ *при* $n>2$ *отрицательные.*

Доказательство. Известно [8], что при узлах Чебышева

$$l_k(\cos\theta)=\frac{1}{n}\left(1+2\sum_{r=1}^{n-1}\cos r\theta_k\cos r\theta\right),\ \theta_k=\frac{(2k-1)\pi}{2n}.$$

Поэтому при узлах (1)

$$A_k^{(n)}=\frac{i_k}{n\sin^2\theta_k},\quad k=1,2,\dots,n \tag{9}$$

где $i_k=\int_0^\pi\left(1+2\sum_{r=1}^{n=1}\cos r\theta_k\cos r\theta\right)\sin^3\theta d\theta$. Продолжим функцию $g(\theta)=\sin^3\theta$, $0\leqq\theta\leqq\pi$ на сегмент $[-\pi,0]$ четным образом и применим к i_k равенство Парсеваля, тогда после простых преобразований, получим, что

$$i_k=\frac{\pi}{2}\left(\frac{8}{3\pi}-\frac{8}{5\pi}\cos 2\theta_k+\frac{6}{\pi}\sum_{s=2}^{m}\left(\frac{1}{4s^2-9}-\frac{1}{4s^2-1}\right)\cos 2s\theta_k\right), \tag{10}$$

где $m=\frac{n-1}{2}$, если n — нечетное и $m=\frac{n-2}{2}$, если n — четное. Из (10) вытекает, что

$$i_k\geqq\frac{1}{2}\left[\frac{16}{15}-\frac{6}{7}+6\left(\frac{1}{15}-\frac{1}{27}\right)+6\left(\frac{1}{35}-\frac{1}{63}\right)+\dots+\frac{6}{4m^2-1}\right].$$

Стало быть, $i_k>0$, $k=1,2,\dots,n$. Поэтому из (9) следует, что $A_k^{(n)}>0$, $k=1,2,\dots,n$. Для крайних чисел $A_0^{(n)}$ и $A_{n+1}^{(n)}$ легко получить равенства

$$A_0^{(n)}=\frac{1}{2}\left(\frac{1}{n+1}-\frac{1}{n-1}\right), \tag{11}$$

при n четном и

$$A_0^{(n)}=\frac{1}{4}\left(\frac{1}{n+2}-\frac{1}{n-2}\right), \tag{12}$$

при n нечетном. Итак, при всех $n>2$, $A_0^{(n)}<0$. Аналогичными рассуждениями, получим что при $n>2$, $A_0^{(n)}=A_{n+1}^{(n)}$. Из теоремы 4 и равенств (11), (12) следует, что при $n>2$ выполняется неравенство

$$\sum_{s=1}^{n}|A_s^{(n)}|\leqq C,$$

ибо $\sum_{k=0}^{n+1}A_k^{(n)}=2$. Поэтому имеет место

Теорема 5. *Интерполяционно-квадратурный процесс, построенный при расширенной матрице узлов Чебышева* (1), *сходится для любой непрерывной функции.*

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CONCERNING BRICKS

By

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0. In [5] the structure of bricks is exposed and it is shown that every brick $\mathscr{B}=(B, *, :, 0)$ arises from some l-group cone by defining on some $[1, c]$

$$a*b := 1 \cup a^{-1}b, \quad b:a := 1 \cup ba^{-1}.$$

Additionally it turned out in [5] that every l-group cone can be extended in a canonical manner to a brick by adjunction of complements.

Hence the theory of bricks proves to be closely related with the theory of l-groups since given a cone $\mathscr{C}$, $\mathscr{C}$ can be extended to an l-group by adjunction of inverses whereas $\mathscr{C}$ can be extended to a brick by adjunction of complements such that in both cases there are a lot of structure invariants (cf. [4]).

Let us consider now a boolean algebra $\mathscr{B}$ and let us regard $\Rightarrow$ as $*$. Then $\mathscr{B}$ satisfies all equations of a symmetric brick (i.e. a brick with $a*b=b:a$) with respect to $\Rightarrow$ and 0 and it was shown in [4] that a boolean algebra considered as implication algebra (cf. [1]) is nothing else but a general brick which satisfies $a*(b*c)=(a*b)*(a*c)$.

Therefore contributions to the theory of bricks are contributions to the theory of l-groups as well as to the theory of boolean algebras and above all theorems of l-group theory have a brick version which concerns boolean algebras, too, and theorems of boolean algebra may have a brick version which concerns the theory of l-groups. Especially it may happen that similar theorems of both theories have a unifying brick version. In spite of this, in general it will not be sufficient to copy a well known result although there are propositions of this type.

The aim of this paper is therefore to show by some selected theorems how brick theory works. For this reason we restrict ourselves to symmetric bricks which, for the sake of convenience, we shall call bricks throughout this paper.

We start by describing bricks as boolean like structures called complemented semigroups here. It will turn out that these semigroups are complementary, too, and hence the theory of bricks can be regarded as a theory of special complementary semigroups.

In the next section some representation theorems are stated which in the boolean case provide Stone's celebrated characterizations of boolean algebras as boolean rings and boolean spaces, respectively.

Complete bricks are investigated in Section 3. Starting from [4] we can give some structure theorems concerning cubes of type $\mathscr{E}^{\omega}$ where $\mathscr{E} := [0,1] \subseteq \mathbf{R}$ with respect to $a*b := \max(0, b-a)$. Finally we then investigate the subbricks of $\mathscr{E}$ which we shall call cubes. Cubes will be characterized in Section 4 by means of ideal theory, of topology and of measure functions such that we obtain classical results concerning l-groups and boolean algebras, respectively.

As a final remark we want to emphasize that the origin of this paper lies in the author's work about divisibility and residuation. Nevertheless, a second approach should be mentioned here. This approach goes back to Iséki's work concerning BCK-algebras. For given a bounded BCK-algebra $\mathcal{B}$, this $\mathcal{B}$ turns out to be a brick (cf. [23, 16]). In spite of this we would like to suggest the short denotion BRICK since it reflects the special situation given in the $\mathbf{R}^3$ where the carrier of a brick forms a "brick" and since according to a funny observation due to A. Day every BRICK-algebra is also a B(RI)CK-algebra.

About the main results of this note the author gave several lectures during the period from 1977 to 1979, among others at the mathematical institutes of the T. U. Munich, the Hungarian Academy of Sciences, the T. U. Vienna and at the Oberwolfach conferences on semigroup theory and universal algebra.

1. Axiomatic aspects

The goal of this section is an investigation of some systems of equations which describe bricks from different points of view.

(1.1) An algebra $\mathcal{B}:=(B,\ *,0)$ is called a brick if it satisfies the axioms:

(BS1) $$(a*a)*b=b$$

(BS2) $$a*(b*c)=b*(a*c)$$

(BS3) $$(a*b)*b=(b*a)*a$$

(BS0) $$0*a=a*a$$

since this definition implies $(a*0)*0=(0*a)*a=(a*a)*a=a$ whence axiom (BR0) of [5] is satisfied.

Thus a brick can be considered as an algebra of type (2,0) and it can be considered as a generalization of Abbot's implication algebra [1]. From this point of view we are led to possibilities of consideration which arise from the facts that a brick carries a natural multiplication and complementation on the one hand, and a natural dualism on the other hand. Thus we can describe a brick as a semigroup with complements as well as a generalized boolean algebra which is done by the theorems below.

(1.2) *Let $\mathcal{B}$ be a brick. Define* $a\circ b:=(b*(a*0))*0$ *and* $a':=a*0$. *Then $\mathcal{B}$ satisfies*

(CS1) $$a\circ(b\circ c)=b\circ(a\circ c)$$

(CS2) $$a\circ a'=0$$

(CS3) $$a\circ 0=0$$

(CS4) $$a\circ 0'=a$$

(CS5) $$a\circ(a\circ b')'=b\circ(a\circ a')'.$$

Proof. (CS1) follows by $a\circ b=(b*(a*0))*0=(a*(b*0))*0=b\circ a$ and

$$a\circ(b\circ c)=a\circ((c*(b*0))*0)=(a*((c*(b*0))*0)*0)*0=(a*(c*(b*0)))*0=$$
$$=(c*(b*(a*0)))*0=c\circ(b\circ a).$$

(CS2) follows from

$$a\circ a'=(a*((a*0)*0))*0=(a*((0*a)*a))*0=(a*a)*0=0.$$

(CS3) follows from

$$a\circ 0=(0*(a*0))*0=(a*(0*0))*0=0.$$

(CS4) follows from

$$a\circ 1=(1*(a*0))*0=(a*0)*0=a.$$

(CS5) follows from

$$a\circ(a\circ b')'=a\circ(((b'*(a*0))*0)*0)=a\circ(a*(b'*0))=a\circ(a*b)=$$
$$=((a*b)*(a*0))*0=((b*a)*(b*0))*0=b\circ(b\circ a')'.$$

Now we show

(1.3) *Let $\mathcal{S}:=(S,\ \cdot,{}',0)$ be an algebra of type (2,1,0) which satisfies the equations* (CS1) *through* (CS5). *Then S forms a symmetric brick under 0 and the operation* $a*b:=(a\circ b')'$.

Proof. Obviously we have $a\circ b=b\circ a$ and putting $0'=:1$ we obtain $0''=1\circ 1'=0$ and hereby

$$a''=1\circ(1\circ a')'=a\circ(a\circ 1')'=a\circ(a\circ 0)'=a\circ 0'=a.$$

This yields (BS1) by

$$(a*a)*b=((a\circ a')'\circ b')'=(0'\circ b')'=b''=b,$$

(BS2) by

$$a*(b*c)=(a\circ(b*c)')'=(a\circ((b\circ c')')')'=$$
$$=(a\circ(b\circ c'))'=(b\circ(a\circ c'))'=b*(a*c),$$

(BS3) by

$$(a*b)*b=((a\circ b')'\circ b')'=((b\circ a')'\circ a')'=(b*a)*a,$$

and axiom (BS0) by

$$0*a=(0\circ a')'=0'=(a\circ a')'=a*a.$$

Let us define now for the sake of typical denotion

(1.4) By a complemented semigroup we mean an algebra $(S,\ \circ,{}',0)$ which satisfies the axioms (CS1) through (CS5).

Then we can state the main result of this section as follows:

(1.5) *Let $\mathcal{B}$ be a brick. Define $a\circ b$ and a' as above. Then B forms a complemented semigroup $\mathcal{C}(\mathcal{B})$ under $\circ$ and $'$. Let $\mathcal{C}$ be a complemented semigroup. Define $*$ and 0 as above. Then C forms a brick $\mathcal{B}(\mathcal{C})$ under 0 and $*$. Finally construct $\mathcal{B}(\mathcal{C}(\mathcal{B}))$ and $\mathcal{C}(\mathcal{B}(\mathcal{C}))$. Then the equations $\mathcal{B}=\mathcal{B}(\mathcal{C}(\mathcal{B}))$ and $\mathcal{C}=\mathcal{C}(\mathcal{B}(\mathcal{C}))$ hold.*

PROOF. All we have to show is the validity of the claimed equations.

(a) Let us start from a brick $\mathscr{B}$. We denote the operations of $\mathscr{B}(\mathscr{C}(\mathscr{B}))$ by $\bar{*}$ and $\bar{0}$. Then we obtain

$$a\,\bar{*}\,b=(a\circ b')'=\big((b'*(a*0))*0\big)'=\big(((b*0)*(a*0))*0\big)'=$$
$$=(b*0)*(a*0)=a*\big((b*0)*0\big)=a*b$$

and $\bar{0}=a\circ a'=(a''*a)'=0$.

(b) Let us start now from a complemented semigroup $\mathscr{C}$. We denote the operations of $\mathscr{C}(\mathscr{B}(\mathscr{C}))$ by $\bar{\circ}$ and $^+$. Then we obtain

$$a\,\bar{\circ}\,b=\big(b*(a*0)\big)*0=\big(b\circ(a\circ 0)'\big)'*0=\big((a*b)'\big)'=a\circ b$$

and $a^+=a*0=(a\circ 0')='a'$.

As was pointed out in [4], the theory of bricks is closely related to the theory of l-groups, since every brick can be canonically embedded into some l-group cone and every l-group cone can be canonically embedded into some brick.

On the other hand, it turned out above that the operator $'$ is involutorial which yields that the definitions $a\triangledown b:=a\circ b$ and $a\vartriangle b:=(a'\circ b')'$ make every brick a selfdual algebra $(B,\triangledown,\vartriangle,')$.

Thus a brick can be considered as a non-idempotent boolean algebra which, by definition, satisfies the laws of de Morgan.

Hence for the sake of completeness it seems to be desirable to have a special name and a typical selfdual system of axioms for these structures, whence we suggest to call them dual bricks and to define them by the description:

(1.6) Let $\mathscr{B}:=(B,\triangledown,\vartriangle,')$ be an algebra of type (2, 2, 1). Then we call $\mathscr{B}$ a dual brick if it satisfies the equations of a brick as well with respect to $\vartriangle$ and $'$ as with respect to $\triangledown$ and $'$ and if, in addition, de Morgan's rules are satisfied.

Obviously by our definition $a\vartriangle a'$ plays the role of the unit with respect to $\triangledown$ and $a\triangledown a'$ plays the role of the unit with respect to $\vartriangle$ since we obtain by de Morgan's laws the equation $a\triangledown(b\vartriangle b')=a\triangledown\big((b\vartriangle b')'\big)'=a\triangledown(b'\triangledown b)'=a$ and the dual hereof.

We finish this section by some remarks concerning complementary semigroups. First of all by (1.5) we have the result that every complemented semigroup is also a complementary semigroup. On the other hand we obtain a generalization of Glivenko's theorem that the closed elements of a bounded brouwerian semilattice form a boolean lattice as is shown by

(1.7) *Let* $(S,\cdot,*)$ *be an abelian complementary semigroup and let* c *be an arbitrary element of* S. *Then the elements* $a_c:=a*c$ *form a symmetric brick under* $*$.

PROOF. We have to show that the set of all a_c is closed under $*$ and that this set satisfies $(a_c*b_c)*b_c=(b_c*a_c)*a_c$. But $a_c*b_c=(a*c)*(b*c)=b(a*c)*c$. Hence all we have to show is the second equation which follows by

$$(a_c*b_c)*b_c=\big((a*c)*(b*c)\big)*(b*c)=$$
$$=\big((a*c)*(((b*c)*c)*c)\big)*\big(((b*c)*c)*c\big)=$$
$$=\big(((b*c)*c)*((a*c)*c)\big)*\big(((b*c)*c)*c\big)=(b_c*a_c)*a_c$$

since the last term function is symmetric in a and b.

2. Concerning representations of bricks

In this section we shall give two representation theorems for arbitrary bricks which in the classical case of boolean algebras coincide with Stone's representation theorems and which are based on well known theorems due to Jaffard and Keimel.

For convenience we shall call PI-ring every commutative ring $\mathscr{R}$ with unit in which all finitely generated ideals are principal ideals. Moreover we shall call divisibility semigroup of $\mathscr{R}$ the semigroup $\mathscr{D}$ of principal ideals of $\mathscr{R}$.

It follows as a first result:

(2.1) *In a PI-integral domain each quotient ideal* $(a):(b)$ *is a principal ideal.*

PROOF. $(a,b)=(d)$ implies $b|ax\Rightarrow b|(a,b)x\Rightarrow b|dx\Rightarrow(b/d)|x$ from which follows $(b):(a)=(b/d)$.

Next we obtain:

(2.2) *Let* $\mathscr{R}$ *be as above and let* c *be different from* 0. *Then the principal ideals of* $\mathscr{R}/(c)=:\mathscr{R}_c$ *form a brick under*

$$(\bar{a})(\bar{b})=(\overline{ab})\quad\textit{and}\quad(\bar{a})'=(\bar{c}):(\bar{a},\bar{c}).$$

PROOF. (i) $(a,c)=(t)\Rightarrow t=ax+cy\Rightarrow\bar{t}=\bar{a}\bar{x}\Rightarrow\bar{t}|\bar{a}\wedge\bar{a}|\bar{t}\Rightarrow(\bar{a})=(\bar{t})$.

(ii) $b|c$ implies

$$\bar{b}|\bar{a}\bar{x}\Rightarrow\bar{b}\bar{y}=\bar{a}\bar{x}\Rightarrow c|ax-by\Rightarrow cu+by=ax\Rightarrow b|ax\Rightarrow\bar{b}|\bar{a}\bar{x}.$$

Hence we obtain $(\bar{b}):(\bar{a})=(b/d)$ whereby the divisibility semigroup $\mathscr{D}_c$ of $\mathscr{R}_c$ forms a complementary semigroup under $\cdot$ and $:$. Moreover we have nearly at once the formula $(\bar{a},\bar{b})=\big((\bar{a}):(\bar{b})\big):(\bar{b})$. Thus $\mathscr{D}_c$ even forms a brick under $\cdot$ and $'$.

Suppose now $\mathscr{B}$ to be a brick and assume $\mathscr{C}(\mathscr{B})$ to be the l-group cone associated with $\mathscr{B}$. Then it is well known by a theorem due to Jaffard that $\mathscr{C}(\mathscr{B})$ can be represented by some abelian l-group cone (cf. [17]). Applying the method of [12] we shall now construct a commutative ring whose divisibility semigroup is isomorphic to $\mathscr{B}$.

(2.3) *Let* $\mathscr{B}$ *be a brick. Then there exists a PI-ring* $\mathscr{R}$ *such that* $(B,\cdot)$ *is a subsemigroup of* $(R,\cdot)$ *and such that* $\mathscr{B}$ *is isomorphic to the divisibility semigroup* $\mathscr{D}$ *of* $\mathscr{R}$.

PROOF. We start from the sums $s=a_1+a_2+\ldots+a_n$ $(a_i\in B)$ considered as elements of the group ring of $\mathscr{G}(\mathscr{C}(\mathscr{B}))$ over the two element field and define $d(\alpha):=a_1\cap a_2\cap\ldots\cap a_n$. Then according to [12] we obtain $d(\alpha)d(\beta)=d(\alpha\beta)$.

We now have to construct a special PI-ring. For this reason we consider the set I of all elements of the form

$$\alpha=a\,\frac{a_1+\ldots+a_n}{b_1+\ldots+b_m}$$

with $a_1\cap a_2\cap\ldots\cap a_n=1=b_1\cap b_2\cap\ldots\cap b_m$ and $a=:d(\alpha)$. We call an ε every element of this kind with $d(\varepsilon)=1$. It follows that that this set is closed under multiplication and addition whence I forms an integral domain in which obviously $(a\varepsilon_a)=(b\varepsilon_b)$ is equivalent to $a=b$. Therefore $\varphi:a\to(a)$ is an isomorphism from $\mathscr{C}(\mathscr{B})$ to the divisibility semigroup of $\mathscr{I}$ and moreover we may assume that B is a subset of I.

We have to show now that $\mathcal{J}$ is a PI-ring. But this follows easily since $(a\varepsilon_a, b\varepsilon_b)= =(a, b)=(a\cap b)\cdot(a*b, b*a)=(a\cap b)(1)$ by construction.

Now we can finish our proof by results obtained above.

Consider $\mathcal{D}_c$ where c is the element 0 of $\mathcal{B}$. Then φ induces an isomorphism between the divisibility semigroup $\mathcal{D}_c$ of $\mathcal{J}/(c)$ and $\mathcal{B}$. For this follows by the facts

(i) $\bar{\varphi}:a\to(\bar{a})$ is a function

(ii) $(\bar{a})=(\overline{a\cap c})$ for all $a\in C(B)$,

since $(\overline{a\cap c})$ has a generator of the form $\bar{a}\bar{x}\mp\bar{y}\bar{c}$ which implies the existence of a generator of the form $\bar{a}\bar{x}$ and hence the equation $(\bar{a})=(\overline{a\cap c})$ as claimed above.

(iii) $\bar{\varphi}$ is a homomorphism

since by (ii) we obtain $(\bar{a})(\bar{b})=(\overline{ab})=(\overline{c\cap ab})=(\overline{ab})$, and

(iv) $\bar{\varphi}$ is bijective

since $ax=b$, $by=a$ implies $c|(ax-b)\wedge c|(by-a)$ whence we obtain $a|b\wedge b|a$ in $\mathcal{J}$ and thereby $a=b$.

As a special case of the last theorem we obtain

(3.4) *Let $\mathcal{B}$ be a boolean algebra and $\mathcal{J}_0$ the ring associated with $\mathcal{B}$ in the sense of* (3.3). *Then $\mathcal{B}$ considered as a lattice is isomorphic to $\mathcal{D}_0$ and $\mathcal{D}_0$ is isomorphic to the subring of idempotents of $\mathcal{J}_0$. Hence $\mathcal{D}_0$ carries an addition which makes D_0 an idempotent ring and the addition of this ring coincides with $(a'\cap b)\cup(b'\cap a)$.*

Proof. All we have to show is that $+$ can be expressed by the boolean operations as claimed above. But obviously we have for idempotent elements of $\mathcal{J}_0$

$$ax=0\Leftrightarrow x(a+1)\Leftrightarrow x\Rightarrow x\geqq a+1.$$

and $a\cap b=a+ab+b$ from which follows

$$a+b=(a\cap b)+(a\cup b)=(a\cap b)\big(1+((b*a)\cup(a*b))\big)=$$

$$=(a\cap b)\big(1+(b'\cap a)\cup(a'\cap b)\big)=(a\cap b)\big((a'\cup b)\cap(b'\cup a)\big)=(a'\cup b)\cap(b'\cup a).$$

Hence it is shown that we obtain the ring of Stone if we endow $(B, \cdot)$ with the addition induced by $(R, \cdot, +)$.

In the second part of this section we consider topological representations. In the classical case of boolean algebras a theorem due to M. H. Stone says that every boolean algebra is associated in a unique manner with a boolean space, i.e. a totally disconnected compact Hausdorff space. This space is constructed on the set of irreducible ideals of $\mathcal{B}$ and the way of construction is folklore today. Moreover, it is well known, too, that a long series of contributions to the theory of topological representation of algebraic structures in general and of partially ordered structures in special was initiated by Stone's papers [21] which led to an extended theory of sheaf representations of algebraic structures (cf. [18], [19]). Thus the situation is pleasant and there is nothing to do but to "copy" Keimel's paper since it is easily seen that its results are consequences of much weaker conditions than those satisfied by an abelian l-group. Therefore we restrict ourselves here to a statement which can be taken from [18] and which we want to formulate here as follows:

Study Sections 2 and 3 of [18]. Then it is clear that the following theorem holds:

(3.5) *Let $\mathcal{B}$ be a brick. Then $\mathcal{B}$ is isomorphic to the brick of all sections of the sheaf $F(\mathcal{B}$, Spec $\mathcal{B})$ over* Spec $\mathcal{B}$ *(where the stalks of this sheaf are totally ordered bricks).*

Finally it is obvious that this representation coincides with Stone's space if $\mathcal{B}$ is boolean since in that case the stalks are two-element algebras whence the sections can be identified with their (clopen) support.

3. Complete bricks

By [4, 5.6] the embedding theorem of [5] and Iwasawa's theorem it is well known that every complete brick is symmetric. Hence it is a fortiori sufficient to speak of a complete instead of a complete symmetric brick.

Complete bricks were already considered by the author in a paper devoted to divisibility semigroups and it turned out that they have a pleasant arithmetic with respect to upper and lower bounds [6]. Hence we may restrict ourselves in this section to the question of representation of complete bricks. We start with a lemma:

(3.1) *Let $\mathcal{B}$ be a brick. Then the idempotents of B form a boolean subalgebra of $\mathcal{B}$ with respect to $\cdot$, $\cap$ and $'$.*

Proof. Let a, b be idempotent. Then ab is equal to $a\cup b$ whereby $(ab)^2= =(a\cup b)^2=a^2\cup ab\cup ba\cup b^2=ab$ and we have next $(a\cap b)^2=a^2\cap ab\cap ba\cap b^2= =a\cap ab\cap ba\cap b=a\cap b$. Let us consider now the complement of a. We obtain

$$a'a'=(a'*a'')'=(a'*a)'=\big((a\cap a')*a\big)'=$$

$$=\big(((a*a')*a')*a\big)'=\big((a'*a')*a\big)'=a'.$$

Obviously the last lemma is true even in arbitrary bricks.

Our next theorem might partially be proven by a theorem of general character due to G. Birkhoff. In spite of this we want to give a complete proof here which is based on an equation which was stated in [6].

(3.2) *Let $\mathcal{B}$ be a complete brick and assume that $a\not\leqq b\wedge b\not\leqq a$ is satisfied. Then one can construct a direct decomposition $\mathcal{U}\times\mathcal{V}$ of $\mathcal{B}$ such that the components of a and b are pairwise comparable.*

Proof. Define

$$U:=\{x|(a*b)\cap x=1\},\quad V:=\{y|\forall x\in U\colon x\cap y=1\}.$$

Then $u:=\bigcup x\ (x\in U)$ satisfies

$$(a*b)*u=(a*b)*x=x=u\quad(x\in U)$$

by the rules of arithmetic which were developed in [6]. Hence $(a*b)\cap u$ is equal to 1 which yields $u\in U$. In an analogous manner we obtain that $v:=\bigcup y\ (y\in V)$ is element of V. Now we consider u^2 and v^2. Here we obtain $(a*b)\cap u^2=1$ from $(a*b)\cap u^2\leqq \leqq\big((a*b)\cap u\big)^2=1$ which means $u^2\in U$, and in a similar manner we get $v^2\in V$. Hence we have $u=u^2$ and $v=v^2$. Up till now we have shown that u and v are disjoint idempotents. On the other hand we have $uv=0$ since from $u*u'=u*(u*0)=u*0=u$ we have $u\cap u'=1$ and thereby $u'\leqq v$ which yields $uv\geqq uu'=0$.

Now we are ready to construct an isomorphism between $\mathscr{B}$ and $(U, *, u)\times\times(V, *, v)$ by

$$\varphi:\ c\to(u\cap c, v\cap c).$$

Obviously φ is a function. Moreover φ is injective in consequence of of the implication

$$u\cap c=u\cap d\wedge v\cap c=v\cap d\Rightarrow c=(u\cap v\cap c)c\cap 0=$$
$$=(u\cap c)(v\cap c)=(u\cap d)(v\cap d)=d.$$

Next φ proves to be surjective since in the case of $x\leqq u\wedge y\leqq v$ we obtain

$$xy\cap u=(x\cup y)\cap u=(x\cap u)\cup(y\cap u)=x$$

and

$$xy\cap v=\ldots \qquad \ldots=y.$$

Hence it remains only to show $\varphi(c)*\varphi(d)=\varphi(c*d)$ the proof of which is obtained subsequently by

(i) $(x*u)*(u*y)=u*y$, since $(x*u)*(u*y)\geqq u*(u*y)=u*y$;

(ii) $u*(x*y)=\big((x*u)*u\big)*\big((x*u)*(u*y)\big)=\big((x*u)*u\big)*\big((u*x)*(u*y)\big)=$
$=(u*x)*\big(((x*u)*u)*(u*y)\big)=(u*x)*(u*y)$;

(iii) $x*(y\cap z)=(x*y)\cap(x*z)$, since $\big(x*(y\cap z)\big)*\big(((x*y)*(x*z))*(x*z)\big)=$
$=\big((x*y)*(x*z)\big)*\big((x*(y\cap z))*(x*z)\big)=$
$=\big((y*x)*(y*z)\big)*\big((y\cap z)*x\big)*(y*z)=1$;

(iv) $u\cap(c*d)=\big((u*c)*(u*d)\big)*(c*d)=\big((c*u)*(c*d)\big)*(c*d)=$
$=(d*u)*(c*u)=(d*u)*\big(((c*u)*u)*u\big)=$
$=\big((cu)u\big)\big((du)u\big)=(u\cap c)*(c\cap d)$.

We go on by

(3.3) Suppose $\mathscr{B}$ to be a brick. Then we denote by $\mathscr{B}(I)$ the boolean subalgebra of all idempotents of $\mathscr{B}$.

It follows as a first structure theorem

(3.4) *Let $\mathscr{B}$ be a complete brick and let p be an atom of $\mathscr{B}(I)$, i.e. let there be no idempotents u with $1<u<p$. Then the interval* $[1, p]$ *is isomorphic to $\mathscr{E}$ or to some brick of type* $\mathscr{S}_n:=(\{0, \ldots, n\}, *)$ *((with* $a*b:=\max(0, b-a)$*).*

Proof. By the proof of (3.3) we know that there are idempotents u, v such that $[1, p]$ considered as complete brick can be decomposed into direct factors if there are two uncomparable elements in $[1, p]$. Hence each pair a, b is comparable which means that $[1, p]$ is of type $\mathscr{E}$ by a theorem due to Clifford [7] or of type $\mathscr{S}_n$.

Obviously the complement of an atom of $\mathscr{B}(I)$ is a hyperatom of $\mathscr{B}(I)$ and our last theorem tells that the ideal generated by p' is maximal. On the other hand assume now that M is a maximal ideal of $\mathscr{B}$. Then $\mathscr{B}/M$ is simple and hence isomorphic to

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kome substructure of $\mathscr{E}$. Hence the natural question arises whether every complete brick can be subdirectly or even directly be decomposed into components of well snown type like those above. Several results in this direction are given below in some representation theorems.

First of all we obtain

(3.5) *Every complete brick has a subdirect decomposition into components of type $\mathscr{E}$ or type $\mathscr{S}_n$.*

Proof. It is sufficient to show that the intersection D of the family of all maximal ideals is equal to $\{1\}$. Hence suppose $c\neq 1$ and $c\in D$. We denote by u the element $\bigcup c^n$ and choose a maximal ideal I which contains u'. I has to contain c but cannot contain u.

Let us consider u. We have $c*(c^k*u)\leqq u$ and $(c^k*u)*c\leqq u$ and since $u^2=u$ by definition and the rules of $*$-arithmetic the interval $[1, u]$ forms a complete brick under $*, u$. Moreover for some l we have $c*(c^l*u)\neq 1$ and $(c^l*u)*c\neq 1$ since by assumption $\mathscr{B}$ is integrally complete and since otherwise $u=c^m\in D$ were satisfied which is impossible since I does not contain u. Hence by (3.4) there are idempotents v, w with the properties $vw=u, v\cap w=1$, and $v\geqq c*(c^l*u)\neq 1$, $w\geqq(c^l*u)*c\neq 1$. Hereby we have $u'vw=0$ with $w\neq 0$ and $w\cap u'v=1$ for some w which yields the existence of a maximal ideal J with the property $u'v\in J$ and thereby $u'\in J$. Hence we obtain

$$c\in J\wedge c*(c^l*u)\in J\Rightarrow c\big(c*(c^l*u)\big)\in J\Rightarrow c\cup(c^l*u)\in J\Rightarrow$$
$$\Rightarrow c^l\in J\wedge c^l*u\in J\Rightarrow c^l(c^l*u)=u\in J$$

which is a contradiction to $u'u=0\notin J$.

(3.6) *Let $\mathscr{B}$ be a complete brick whose boolean subalgebra $\mathscr{B}(I)$ is atomic. Then $\mathscr{B}$ is a direct product of type $\mathscr{E}^\omega$.*

Proof. Let $p_i\,(i\in I)$ be the atoms of $\mathscr{B}(I)$. Then we can state

(i) $\forall a\colon a=a\cap 0=\bigcup(a\cap p_i)\quad(i\in I)$

(ii) $a\neq b\Rightarrow\exists j\in I\colon a\cap p_j\neq b\cap p_j$

(iii) $x_i\leqq p_i\Rightarrow\bigcup x_i=\bigcup\big((\bigcup x_i)\cap p_i\big)\quad(i\in I)$.

Hence there is a bijection φ between B and the product ΠP_i where P_i is equal to $\{x|x\leqq p_i\}$ $(i\in I)$ if we define

$$\varphi(a):=\{(i, p_i\cap a)|i\in I\}.$$

Moreover we have obviously

(iv) $(a\cap 0)*(b\cap 0)=(a*b)\cap 0$

which implies

$$\big(\bigcup(a\cap p_i)\big)*\big(\bigcup(b\cap p_i)\big)=\bigcup\big((a*b)\cap p_i\big),$$

i.e. $\varphi(a)*\varphi(b)=\varphi(a*b)$. Thus our proof is complete by (3.4).

As a specialization of (3.6) and as a generalization of a theorem due to Tarski [22] we obtain

(3.7) *A brick is isomorphic to some real cube iff it is complete, totally distributive and atomless.*

PROOF. According to Tarski's method we form all intersections $\bigcap u_i\ (i\in I)$ with the property that all u_i are idempotent and such that exactly one element of each pair u_i, u_i' is selected. Then 0 is the union of all such intersections and all intersections are equal to 1 or equal to some atom of $\mathscr{B}(I)$. Hence our proof is complete by (3.6) since $\mathscr{B}$ is atomless and since the necessity of our condition is obvious.

Replacing atomless by atomic we obtain by (3.4)

(3.8) *A brick is isomorphic to some lattice brick iff it is complete, totally distributive and atomic.*

In the last part of this section we shall be concerned with the special cubes $\mathscr{E}^n$. Here we succeed in describing by notions of general topology.

(3.9) *A brick is a cube $\mathscr{E}^n$, considered with respect to the natural brick operations iff $(B, \leqq)$ is compact in the interval topology, each maximal ideal is closed in this topology, and each minimal ideal is connected in this topology.*

PROOF. Obviously our condition is necessary. Hence assume that $\mathscr{B}$ satisfies the required properties.

By a theorem due to O. FRINK [11] $(B, \leqq)$ is compact in the interval topology iff $(B, \leqq)$ is complete. Therefore $\mathscr{B}$ is complete.

Next let I be a maximal ideal. Since I is closed, it has a maximum or can be covered by finitely many sets of type $B-[x, 0]$ where x lies in I. Hence I has a maximum u since otherwise the second assumption leads to the contradiction that some $x_1\cup x_2\cup\ldots\cup x_n$ is not covered although it is an element of I.

We consider the maximum u of I. It is idempotent and has an (idempotent) complement which is atom in $\mathscr{B}(I)$. Vice versa, every hyperatom of $\mathscr{B}(I)$ is maximum of a maximal ideal of $\mathscr{B}$. We show: There are only finitely many atoms in $\mathscr{B}(I)$. This follows from the fact that otherwise the set of elements of B which are divided by at most finitely many atoms is a subset of some maximal ideal $[1, u]$ whence the atom u' has to be a divisor of u, a contradiction to $u'\nleqq u$.

Hence there are only finitely many atoms p_i in $\mathscr{B}(I)$ and $0=\bigcup p_i$ since otherwise there were a maximal u in $\mathscr{B}(I)-\{0\}$ the complement of which were an atom but did not divide $\bigcup p_i$. Hereby the proof is complete in consequence of our third assumption and (3.4).

In analogy to the above situation we get once more an opposite theorem.

(3.10) *A brick $\mathscr{B}$ is a finite brick iff $(B, \leqq)$ is compact in the interval topology, each maximal ideal is closed in this topology, and each minimal ideal is totally disconnected in this topology.*

4. Cube algebras

In Section 3 it turned out that complete bricks have pleasant representations by substructures of $\mathscr{E}$-cubes. In this section we ask in general which bricks are embeddable into some $\mathscr{E}^\omega$. Obviously these bricks satisfy $\forall a\in B\exists n\in \mathrm{N}: (a^n)'\nleqq a$ and in connection with problems arising in the theory of divisibility semigroups it was shown in [6] that this property is also sufficient for a brick to have a complete brick extension. Hence defining

(4.1) A brick is called a cube algebra or shortly a cube if it is subbrick of some $\mathscr{E}^\omega$; and

(4.2) A brick is called integrally closed if it has the property $\forall a\in B\exists n\in \mathrm{N}$: $(a^n)'\nleqq a$;

we obtain as consequences of [6] and (3.5) the following theorems:

(4.3) *A brick is a cube iff it is integrally closed.*

(4.4) *A brick is a cube if it has a complete brick extension.*

Applying (4.4) and [4, 3] to special structures we get Dedekind's theorem that the ordered group of rationals can be completed by cuts to a (conditionally) complete totally ordered group (called the group of reals) and the Glivenko—Stone theorem that the completion by cuts of a boolean algebra is again a boolean algebra (cf. [1]).

Let us consider next cube algebras from the topological point of view. By methods introduced by M. H. Stone it is routine today to construct an algebra of continuous functions from a compact Hausdorff space to the unit interval which contains a given brick as a substructure once one has some decomposition of this cube into subdirect factors contained in $\mathscr{E}$. In spite of this we shall give the main ideas here for the sake of selfcontainedness.

In a first step we endow $\mathbf{E}^B$ with the product topology and get thus a compact Hausdorff space $\mathbf{H}$.

Next we consider the set of all homomorphisms from $\mathscr{B}$ to $\mathscr{E}$. This set is a closed subset of $\mathbf{E}^B$ and hence a compact Hausdorff space with respect to the relative topology which follows from the properties of product topology. Denote this space by $\mathbf{D}$. Obviously the set of all continuous functions φ from $\mathbf{D}$ to $\mathbf{E}$ endowed with its natural topology forms an integrally closed brick under pointwise residuation. We define

$$\hat{a}\colon \mathbf{D}\to\mathbf{E} \quad\text{by}\quad \hat{a}(\varphi)=\varphi(a).$$

Then $a\to\hat{a}$ provides a monomorphism from $\mathscr{B}$ to the brick of all continuous functions from $\mathbf{D}$ to $\mathbf{E}$ since $a\neq b$ implies $\hat{a}\neq\hat{b}$ in consequence of (3.4) and since $(a*b)\hat{}\,(\varphi)=$ $=\varphi(a*b)=\varphi(a)*\varphi(b)=\hat{a}(\varphi)*\hat{b}(\varphi)$.

Hence on the basis of Tychonoff's pioneer paper about product spaces and by applying representation ideas due to M. H. Stone we obtain from (3.4) the following theorem.

(4.5) *A brick is a cube iff it is subbrick of the brick of all continuous functions from some compact Hausdorff space to the real unit interval space.**

Recall that every brick has a sheaf representation the stalks of which are subdirectly irreducible totally ordered bricks. In the special case of cubes we obtain as another specialization of Keimel's general representation theorem:

(4.6) *A brick $\mathscr{B}$ is a cube iff it has a sheaf representation over the set of all maximal ideals of $\mathscr{B}$ by stalks which are substructures of $\mathscr{E}$.*

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* Obviously this theorem is closely related to Fan's Theorem concerning the respesentation of archimedean vector lattices [9]. See also FLEISCHER [10].

For this follows from (3.5).

In the remainder of this section we shall be concerned with measure functions on bricks. To have short denotions we symbolyze by $\mathscr{B}$ an arbitrary brick, by $\mathscr{P}$ the product extension of this brick and by $\mathscr{G}$ the lattice ordered quotient group of $\mathscr{P}$. Recall once more that cone and group are abelian iff $\mathscr{B}$ is symmetric. We repeat

(4.6) Let $\mathscr{B}$ be a brick. By a measure function of $\mathscr{B}$ we mean a non-negative real valued function $\mu: a \to |a|$ which satisfies $|a|+|a*b|=|b|+|b*a|$ and $|1|=0$.

Obviously in a symmetric brick we have $|(a*b)*b|+|a*b|=|b|$ and $|a|+ +|a*(a\cup b)|=|a\cup b|$ from which follows that μ is a valuation of $(B, \cap, \cup)$.

Special measures of $\mathscr{B}$ are the homomorphisms from $\mathscr{B}$ to $\mathscr{E}$. This leads us to a first characterization theorem for cube algebras by means of measure.

(4.7) *A brick is a cube iff for every element a of B there is a measure μ such that $\mu(a)=1$.*

Proof. If there is a measure for each $a\neq 1$ such that a takes the value 1, $\mathscr{B}$ has to be integrally closed since otherwise 0 could not take a real value.

Assume now that for every $a\neq 1$ of B there is a homomorphism h from $\mathscr{B}$ to $\mathscr{E}$ such that $h(a)\neq 0\in \mathbf{R}$. Then we obtain a measure μ with $\mu(a)\neq 1$ by multiplying all $h(b)$ $(b\in B)$ with the factor $1/h(a)$.

Before we are ready to prove a second characterization theorem by means of measure we have to give a further definition.

(4.8) Let A be a subset of the carrier P of $\mathscr{P}$ and let f be a function from A to $\mathbf{R}^{\geqq 0}$. Then we call f a partial valuation of $\mathscr{B}$ if the canonical extension of f to the subsemigroup $\mathscr{A}$ of $\mathscr{P}$ which is generated by A, is an isotone valuation of $\mathscr{A}$, i.e. satisfies $f(a)+f(b)=f(ab)$.

Obviously $f(1)$ has to be equal to 0 in consequence of $f(1)=f(1)+f(1)$ and we know by [4, 5.4] that every measure function on a symmetric brick has a unique extension to the corresponding abelian l-group.

The set of partial valuations of an l-group is in no case empty and it satisfies the conditions of Zorn's lemma. Hence in any case there is a maximal partial valuation.

(4.9) *Let f be a maximal partial valuation of the abelian l-group cone $\mathscr{P}$. Then the domain $D(f)$ is closed with respect to multiplication and convex with respect to $\leqq$.*

Proof. $D(f)$ is closed with respect to multiplication by definition. Moreover $D(f)$ contains 1 since otherwise $f(1):=0$ would yield an extension.

Let us suppose now that a and b satisfy $1\leqq a\leqq b\in D\wedge a\notin D$. Then the set $S(a)$ of non-negative real numbers which satisfy

$$c,\, d\in D(f)\wedge c\leqq a^p d \Rightarrow \frac{f(c)-f(d)}{p}\leqq s$$

and

$$u,\, v\in D(f)\wedge a^q u\leqq v \Rightarrow s\leqq \frac{f(v)-f(u)}{q}$$

is not empty since

$$c\leqq a^p d\wedge a^q u\leqq v\Rightarrow (cd^{-1})^q\leqq a^{pq}\leqq (vu^{-1})^p\Rightarrow c^q d^{-q}\leqq v^p u^{-p}\Rightarrow$$

$$\Rightarrow c^q u^p\leqq v^p d^q\Rightarrow q\cdot f(c)-q\cdot f(d)\leqq p\cdot f(v)-p\cdot f(u)\Rightarrow \frac{f(c)-f(d)}{p}\leqq \frac{f(v)-f(u)}{q}.$$

Hence we can choose an element $|a|$ of $S(a)$ and define a function $\|$ by

$$|x| := \begin{cases} f(x) & \text{for}\quad x\in D \\ |a| & \text{for}\quad x=a. \end{cases}$$

This produces an extension of f which is shown by a proof of the implication

$$a^m c\leqq a^n d\Rightarrow m\cdot|a|+f(c)\leqq n\cdot|a|+f(d)$$

since hereby $|a^q c|:=p|a|+f(c)$ will turn out to be a function and moreover to be a valuation. In order to prove the implication stated above we consider several cases.

Obviously the implication holds if m is equal to n.

Let us suppose now $m<n$. Then putting $n-m=:p$ we obtain

$$a^m c\leqq a^n d\Rightarrow \frac{f(c)-f(d)}{p}\leqq |a|\Rightarrow$$

$$\Rightarrow f(c)\leqq p\cdot|a|+f(d)\Rightarrow m\cdot|a|+f(c)\leqq n\cdot|a|+f(d).$$

Finally we succeed in a similar manner if $m>n$. Hence f could be extended if there were an element a with the property $1\leqq a\leqq b\in D\wedge a\notin D$. This completes the proof.

As an immediate consequence of the last theorem we state

(4.10) *A brick $\mathscr{B}$ is a cube iff for every $D=\{a(\neq 1), 0\}$ there exists a positive valuation.*

Applying this last theorem we obtain as an improvement of a result concerning boolean algebras

(4.11) *Let $\mathscr{B}$ be a brick and $\mathscr{U}$ be a subbrick of $\mathscr{B}$. Then every measure function of $\mathscr{U}$ can be extended to a measure function of $\mathscr{B}$.*

Proof. Since 0 is element of U a measure function of $\mathscr{U}$ can be extended to a measure function of $\mathscr{B}$ iff in $\mathscr{P}$ the implication

$$(*)\qquad \prod_1^n a_\nu\leqq \prod_1^n b_\nu\Rightarrow \sum_1^n |a_\nu|\leqq \sum_1^n |b_\nu|$$

holds and this is a consequence of [4, 5.4].

Theorem (4.10) is closely related to a theorem due to Horn and Tarski concerning measures of boolean algebras. Hence it seems desirable to have a direct proof of the equivalence of the Tarski Horn condition on the one hand and our condition $(*)$ on the other hand. Recall that $(*)$ could be formulated also in $\mathscr{B}$, since $a=b\Leftrightarrow \Leftrightarrow(a*b)(b*a)=1$, $ab*c=b*(a*c)$ and $a*bc=(a*b)((b*a)*c)$.

We start with the following lemma:

(4.12) *Let $\mathscr{B}$ be a symmetric brick and n a natural number. Then the equation*

$$\prod_1^n a_\nu = \prod_1^n \sigma_k(a_1, \dots, a_n)$$

holds where σ_k is the LUB of all intersections of exactly k factors of $a_1 \cdot a_2 \cdot \dots \cdot a_n$.

Proof. Notice that one has to differentiate between a_i and a_j if $i \neq j$ also in the case of $a_i = a_j$. Obviously the formula under consideration holds for $n=1$. Moreover it holds for $n=2$ by $ab=(a \cup b)(a \cap b)$. Let now $n=3$. Then the following calculation is symptomatic for induction:

$$\begin{aligned} a_1 a_2 a_3 &= (a_1 \cup a_2)(a_1 \cap a_2) a_3 = (a_1 \cup a_2 \cup a_3)((a_1 \cup a_2) \cap a_3)(a_1 \cap a_2) = \\ &= (a_1 \cup a_2 \cup a_3)((a_1 \cap a_3) \cup (a_2 \cap a_3))(a_1 \cap a_2) = \\ &= (a_1 \cup a_2 \cup a_3)((a_1 \cap a_3) \cup (a_2 \cap a_3) \cup (a_1 \cap a_2))(a_1 \cap a_2 \cap a_3) = \\ &= \sigma_1(a_1, a_2, a_3) \cdot \sigma_2(a_1, a_2, a_3) \cdot \sigma_3(a_1, a_2, a_3). \end{aligned}$$

Recall that the last lemma is true for boolean algebras as well as for abelian l-group cones (since they have a symmetric brick extension).

As an immediate consequence of our last lemma we obtain that the product $\prod_1^n a_\nu$ is equal to or less than $\prod_1^n b_\nu$ if after filling up the products by factors 1 such that they are of equal length l all σ_k $(1 \leqq k \leqq l)$ with respect to $(a_1, \dots, a_l)$ are equal to or less than the corresponding σ_k with respect to $(b_1, \dots, b_l)$.

Assume now that we are given a boolean algebra. Then we can prove the converse, too, i.e.

(4.13) *Let $\mathscr{B}$ be boolean and let $a_1, \dots, a_l, b_1, \dots, b_l$ be elements of B which satisfy in $\mathscr{P}$ $a_1 \cdot a_2 \cdot \dots \cdot a_l \leqq b_1 \cdot b_2 \cdot \dots \cdot b_l$. Then for every $k \leqq l$ we can verify the relation*

$$\sigma_k(a_1, \dots, a_l) \leqq \sigma_k(b_1, \dots, b_l).$$

Proof. First we obtain

$$\text{(i)} \qquad a_1 \leqq b_1 \cup \dots \cup b_l$$

since we have $a_1 \leqq b_1 \cdot b_2 \cdot \dots \cdot b_n$ in $\mathscr{P}$ and

$$a_1 \leqq c_1 c_2 \Rightarrow (c_1 \cap a_1) \leqq c_1 \wedge (c_1 * a_1) \leqq c_2,$$

and thereby

$$a_1 \leqq c_1 c_2 \Rightarrow a_1 = (c_1 \cap a_1)(c_1 * a_1) \leqq (c_1 \cap a_1) \cup (c_1 * a_1) \leqq c_1 \cup c_2$$

from which by way of induction we are led to (i) if we consider $b_1 b_2 \dots b_n$ as $(b_1 \cdot b_2 \cdot \dots \cdot b_{n-1}) b_n$. Hence our statement is true for $k=1$. Suppose now that our proposition is true for all i with $1 \leqq i \leqq k$, with respect to all $m \geqq k$. We consider first the special case of

$$c_1 \cdot \dots \cdot c_k c_{k+1} \leqq d_1 \cdot \dots \cdot d_k d_{k+1}$$

and denote by $\sigma_i(c)$ the element $\sigma_i(c_1, \dots, c_{k+1})$, by $\sigma_i(d)$ the element $\sigma_i(d_1, \dots, d_{k+1})$.

It will turn out that $\sigma_{k+1}(c) \leqq \sigma_{k+1}(d)$, i.e.

$$c_1 \cap \dots \cap c_{k+1} \leqq d_1 \cap \dots \cap d_{k+1}$$

since by assumption $\sigma_i(c) \leqq \sigma_i(d)$ for all i with $1 \leqq i \leqq k$ whereby

$$\left(\prod_1^k \sigma_i(c)\right) \sigma_{k+1}(c) \leqq \left(\prod_1^k \sigma_i(d)\right) \sigma_{k+1}(d) = \left(\prod_1^k \sigma_i(c)\right)\left(\prod_1^k (\sigma_i(c) * \sigma_i(d))\right) \sigma_{k+1}(d) \Rightarrow$$

$$\Rightarrow \sigma_{k+1}(c) \leqq \left(\prod_1^k (\sigma_i(c) * \sigma_i(d))\right) \sigma_{k+1}(d)$$

and since

$$\sigma_{k+1}(c) \leqq \sigma_i(c) \quad (1 \leqq i \leqq k) \Rightarrow \sigma_{k+1}(c) \cap (\sigma_i(c) * \sigma_i(d)) = 1$$

whence we succeed by combining these two results.

Now we are able to complete the proof since we obtain $\sigma_{k+1}(a) \leqq \sigma_{k+1}(b)$ by (ii) $x \cap yz \leqq (x \cap y)(x \cap z)$ and formula (i) from the implication

$$a_{l,1} \cap \dots \cap a_{l,k+1} \leqq b_1 \cap b_2 \cap \dots \cap (b_{k+1} b_{k+2} \dots b_m) \leqq \prod_1^{m-k} ((b_1 \cap \dots \cap b_k) \cap b_{k+\nu}) \Rightarrow$$

$$\Rightarrow a_{l,1} \cap \dots \cap a_{l,k+1} \leqq \bigcup_1^{m-k} ((b_1 \cap \dots \cap b_k) \cap b_{k+\nu}) \leqq \sigma_{k+1}(b).$$

Applying (4.12) and (4.13) we can now prove the Horn—Tarski theorem

(4.14) *Let $\mathscr{B}$ be a boolean algebra and A be a subset of B with $0 \in A$. Then a real valued function f on A can be extended to a measure μ of $\mathscr{B}$ iff for every pair of n-tuples $(a_1, \dots, a_n)$, $(b_1, \dots, b_n)$ the implication*

$$(*) \qquad (\forall i (1 \leqq i \leqq n) \colon \sigma_i(a) \leqq \sigma_i(b)) \Rightarrow \sum f(a_i) \leqq \sum f(b_i)$$

holds.

Proof. Suppose that f has an extension μ and that the left side of $(*)$ is true. Then we obtain

$$\sum \mu(a_i) = \sum \mu(\sigma_i(a)) \leqq \sum \mu(\sigma_i(b)) = \sum \mu(b_i).$$

Suppose now that $(*)$ is satisfied. Then the implication

$$\prod a_i \leqq \prod b_i \Rightarrow \sigma_i(a) \leqq \sigma_i(b) \ (\forall i) \Rightarrow \sum f(a_i) \leqq \sum f(b_i)$$

holds.

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THE BESSEL POLYNOMIAL MOMENT PROBLEM[*]

By

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1. Introduction. Since 1949 when the Bessel polynomials appeared [2] a hunt has been on to find a weight function ψ, defined on the real axis, with respect to which the polynomials would be orthogonal. Although at least two devices have been found which formally seem to fill the role of orthogonality generation, none was expressible in terms of integration on the real axis.

The purpose of this paper is to show that the Bessel polynomials $y_n(x, a, b)$ are orthogonal with respect to the distribution ψ, given by

$$\langle f, \psi\rangle = \lim_{\varepsilon\to 0}\frac{1}{\pi}\int_{\alpha}^{\beta} f(x)\,\mathrm{Im}\,\{(-b/z)_1F_1[1, a, -b/z]\}\,dx$$

where $z=x+i\varepsilon$, and $\alpha<0$, $\beta>0$, but are otherwise arbitrary.

The key to showing that the above formula is valid is through the moments associated with the Bessel polynomials

$$\mu_n = (-b)^{n+1}/(a)_n, \quad n = 0, 1, \dots,$$

where $(a)_n=a(a+1)\dots(a+n-1)$, which were discovered in [2] by examining the differential equation satisfied by the Bessel polynomials. It is well known that, given the moments μ_n, there exists a function of a complex variable $f(z)$, called the Cauchy representation, given by

$$f(z) = \sum_{n=0}^{\infty}\mu_n/z^{n+1}$$

when the support of the weight function is compact, and in general satisfying

$$\lim_{z\to\infty} z^{k+1}\left[f(z)+\sum_{n=0}^{k}\mu_n/z^{n+1}\right] = -\mu_k$$

when for some fixed δ, $0<\delta<\pi/2$, $\delta<\arg z<\pi-\delta$.

For the Bessel polynomials it is easy to see that

$$f(z) = \sum_{n=0}^{\infty}(-b/z)^{n+1}/(a)_n = (-b/z)_1F_1[1, a, -b/z].$$

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By using the Stieltjes—Perron formula ([4] or in the distributional case [1]), the formula for ψ immediately follows. The fact that the interval $[\alpha, \beta]$ needs only to include the point 0 follows from the analyticity of $f(z)$ in any region excluding the origin.

2. Results. We state in summary

THEOREM 1. *The Bessel polynomials are mutually orthogonal with respect to* ψ. *That is, if* $n \neq m$

$$\langle y_n y_m, \psi \rangle = 0.$$

This, of course, follows from the Stieltjes—Perron formula, which shows that ψ generates the moments through $\langle x^n, \psi \rangle = \mu_n$, $n=0, 1, \ldots$. A direct proof may be quickly given, however by noting that

$$\psi(\beta)-\psi(\alpha) = \lim_{\varepsilon\to 0} \frac{1}{\pi} \int_\alpha^\beta \sum_{n=0}^{\infty} \frac{(-b)^{n+1}}{(a)_n} \sum_{j=0}^{\left[\frac{n}{2}\right]} \binom{n+1}{2j+1} \frac{(-1)^{j+1}\varepsilon^{2j+1}x^{n-2j}}{(x^2+\varepsilon^2)^{n+1}}\, dx.$$

The term within the brackets is $(-1)^n \frac{d^n}{dx^n}\left[\frac{\varepsilon}{x^2+\varepsilon^2}\right]/n!$. As $\varepsilon \to 0$ it approaches $(-1)^n \pi\, \delta^{(n)}(x)/n!$, where $\delta^{(n)}(x)$ is the n^{th} distributional derivative of the Dirac delta function. Thus ψ has the distributional expansion

$$\psi = \sum_{n=0}^{\infty} (-1)^n \mu_n \delta^{(n)}(x)/n!$$

which is well known [3].

THEOREM 2. *For* $n=0, 1, \ldots,$

$$\langle y_n^2, \psi \rangle = (-1)^{n+1} b n!/(2n+a-1)(a)_{n-1}.$$

This follows from using the Bessel polynomial's three term recurrence relation [2] and the knowledge that $\mu_0 = -b$.

Special cases. The instances where $a=b=2$ and $a=b=1$ are of historical importance. When $a=b=2$, the moments $\mu_n = (-2)^{n+1}/(n+1)!$. ψ is given by

$$\psi(\beta)-\psi(\alpha) = \lim_{\varepsilon\to 0} \frac{-1}{\pi} \int_\alpha^\beta \exp\left[\frac{-2x}{x^2+\varepsilon^2}\right] \sin\left[\frac{2\varepsilon}{x^2+\varepsilon^2}\right] dx.$$

Then

$$\langle y_n y_m, \psi \rangle = \begin{cases} 0 & \text{if } n \neq m, \\ (-1)^{n+1} 2/(2n+1) & \text{if } n = m. \end{cases}$$

When $a=b=1$, the moments $\mu_n = (-1)^{n+1}/n! \ldots$ ψ is given by

$$\psi(\beta)-\psi(\alpha) = \lim_{\varepsilon\to 0} \frac{1}{\pi} \int_\alpha^\beta \exp\left[\frac{-x}{x^2+\varepsilon^2}\right] \left\{\frac{-x}{x^2+\varepsilon^2} \sin\left[\frac{\varepsilon}{x^2+\varepsilon^2}\right] + \frac{\varepsilon}{x^2+\varepsilon^2} \cos\left[\frac{\varepsilon}{x^2+\varepsilon^2}\right]\right\} dx.$$

Then

$$\langle y_n y_m, \psi \rangle = \begin{cases} 0 & \text{if } n \neq m, \\ (-1)^{n+1}/2 & \text{if } n = m. \end{cases}$$

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ON DECOMPOSABILITY OF CURVATURE TENSOR OF A FINSLER MANIFOLD

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1. Introduction

Studying recurrent affine motion in a recurrent non-Riemannian manifold, K. TAKANO [13] observed that the curvature tensor B^i_{jkh} is decomposable in the form $B^i_{jkh}=X^i\varphi_{jkh}$, where X^i and φ_{jkh} are non-null vector and tensor respectively. In another paper [14], he studied a recurrent non-Riemannian manifold whose curvature tensor is decomposable in the above form. Following K. Takano, B. B. SINHA and S. P. SINGH [11], and RAM HIT [2] studied a recurrent Finsler manifold whose curvature tensor H^i_{jkh} is decomposable in the form $H^i_{jkh}=X^i\varphi_{jkh}$. B. B. SINHA and S.P. SINGH [12], RAM HIT [2], and H. D. PANDE and H. S. SHUKLA [7] studied a recurrent Finsler manifold whose curvature tensor is decomposable in the form $H^i_{jkh}=\dot{x}^i\varphi_{jkh}$. B. B. SINHA [10] also studied such decomposition in a birecurrent Finsler manifold. H. D. PANDE with T. A. KHAN [6] considered a recurrent Finsler manifold whose curvature tensor is decomposable in the form $H^i_{jkh}=X^i_j\varphi_{kh}$, while he assumed another decomposition of the form $H^i_{jkh}=P_jX^i_{kh}$ in his paper [7] jointly with H. S. SHUKLA. A natural question arises in this context whether such decompositions of curvature tensor of a Finsler manifold are possible. Partial answer of this question is given by the author in his paper [8]. The aim of this paper is to discuss the above problem.

2. Preliminaries

Let F_n be an n-dimensional Finsler manifold endowed with a metric function[1] F satisfying the requisite conditions [9]. Let the components of the metric tensor, Berwald's connection and Berwald's curvature tensor be denoted by g_{ij}, G^i_{jk} and H^i_{jkh} respectively. The curvature tensor H^i_{jkh} is skew-symmetric in k and h, and is positively homogeneous of degree zero in $\dot{x}^i$'s. Transvections of the curvature tensor by $\dot{x}$'s and contractions of indices yield the following:

$$(2.1)\qquad \begin{cases} \text{a)}\ H^i_{kh}=\dot{x}^jH^i_{jkh}, & \text{b)}\ H^i_h=\dot{x}^kH^i_{kh}, \\ \text{c)}\ H_{jk}=H^i_{jki}, & \text{d)}\ H_k=H^i_{ki}, \qquad \text{e)}\ (n-1)H=H^i_i. \end{cases}$$

The above tensors are also related by

$$(2.2)\qquad \begin{cases} \text{a)}\ H^i_{jkh}=\dot{\partial}_jH^i_{kh},{}^{2} & \text{b)}\ H^i_{kh}=\dfrac{1}{3}(\dot{\partial}_kH^i_h-\dot{\partial}_hH^i_k), \\ \text{c)}\ \dot{x}^kH_{kh}=H_h, & \text{d)}\ H_{kh}=\dot{\partial}_kH_h, \quad \text{e)}\ H_h\dot{x}^h=(n-1)H. \end{cases}$$

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[1] Unless stated otherwise all the entities are considered as functions of line-elements $(x^i, \dot{x}^i)$. The indices $i, j, k, \ldots$ assume positive integral values $1, 2, 3, \ldots, n$.

[2] $\dot{\partial}_i\equiv\partial/\partial\dot{x}^i$.

The vector $y_i \overset{\text{def}}{=} g_{ij}\dot{x}^j$ satisfies the following:

$$\text{(2.3)} \qquad \text{a) } y_i\dot{x}^i = F^2, \quad \text{b) } \dot{\partial}_i y_j = g_{ij}, \quad \text{c) } \frac{y_i}{F} = \dot{\partial}_i F.$$

It is well known that the metric function F is a covariant constant, i. e.

$$\text{(2.4)} \qquad \mathscr{B}_k F = 0,$$

where $\mathscr{B}_k$ is Berwald's covariant differential operator. Differentiating (2.4) covariantly with respect to x^j and utilizing the commutation formula exhibited by

$$\mathscr{B}_j\mathscr{B}_k\varphi - \mathscr{B}_k\mathscr{B}_j\varphi = -(\dot{\partial}_r\varphi)H^r_{kj},$$

we get

$$\text{(2.5)} \qquad (\dot{\partial}_r F)\, H^r_{jk} = 0;$$

which, in view of (2.3c), may be written as

$$\text{(2.6)} \qquad y_r H^r_{jk} = 0.$$

Transvecting (2.6) by $\dot{x}^j$ and using (2.1b), we get

$$\text{(2.7)} \qquad y_r H^r_k = 0.$$

Differentiating (2.7) partially with respect to $\dot{x}^j$ and using (2.3b), we get

$$\text{(2.8)} \qquad H_{k..j} + y_r\dot{\partial}_j H^r_k = 0,$$

where $H_{k..j} \overset{\text{def}}{=} g_{rj}H^r_k$. Taking skew-symmetric part of (2.8) and using (2.2b) and (2.6), wec onclude

$$\text{(2.9)} \qquad H_{k..j} = H_{j..k}.$$

A Finsler manifold is called flat if and only if the curvature tensor vanishes.

3. Different types of decompositions

Let us consider a Finsler manifold whose curvature tensor is decomposable, then the possibilities of its form are as follows:

$$\text{(3.1)} \qquad \text{a) } H^i_{jkh} = X^i\varphi_{jkh}, \quad \text{b) } H^i_{jkh} = P_j\psi^i_{kh}, \quad \text{c) } H^i_{jkh} = T^i_j Y_{kh}.$$

We shall discuss these possibilities one by one. For this discussion we need:

Lemma 3.1. *A Finsler manifold admitting*

$$\text{(3.2)} \qquad y_i H^i_{jkh} = 0$$

is flat.

Proof. Let us consider a Finsler manifold admitting (3.2). Differentiating (2.6) partially with respect to $\dot{x}^j$ and using (2.2a), (2.3b) and (3.2) we have

$$\text{(3.3)} \qquad g_{ij}H^i_{kh} = 0.$$

Transvecting (3.3) by g^{jm} and using $g^{jm}g_{ij} = \delta^m_i$, we get $H^m_{kh} = 0$; which, in view of (2.2a), implies $H^i_{jkh} = 0$. Thus, the manifold considered is flat.

Theorem 3.1. *The curvature tensor of a non-flat Finsler manifold cannot be decomposable in the form* (3.1a).

Proof. Let us consider a non-flat Finsler manifold whose curvature tensor is decomposable in the form (3.1a). Transvecting (3.1a) by $\dot{x}^j$ and using (2.1a), we get

$$\text{(3.4)} \qquad H^i_{kh} = X^i\varphi_{kh},$$

where $\varphi_{kh} = \dot{x}^j\varphi_{jkh}$. Transvecting (3.4) by y_i and using (2.6) we have $(y_iX^i)\,\varphi_{kh} = 0$; which implies at least one of the conditions:

$$\text{(3.5)} \qquad \text{a) } \varphi_{kh} = 0, \qquad \text{b) } y_iX^i = 0.$$

The condition (3.5a) and (3.4) give $H^i_{kh} = 0$; which, in view of (2.2a), implies $H^i_{jkh} = 0$. Thus, we get a contradiction. Transvecting (3.1a) by y_i and using (3.5b), we get (3.2); which, in view of Lemma (3.1), implies that the manifold considered is flat. Thus we see that either of the conditions (3.5a) and (3.5b) leads to a contradiction. This completes the proof.

Theorem 3.2. *The curvature tensor of a non-flat Finsler manifold cannot be decomposable in the form* (3.1b).

Proof. Let us consider a non-flat Finsler manifold whose curvature tensor is decomposable in the form (3.1b). Transvecting (3.1b) by $\dot{x}^j$ and using (2.1a), we have

$$\text{(3.6)} \qquad H^i_{kh} = P\psi^i_{kh},$$

where $P = P_j\dot{x}^j$. Transvecting (3.6) by y_i and using (2.6) we get $Py_i\psi^i_{kh} = 0$; which implies at least one of the conditions:

$$\text{(3.7)} \qquad \text{a) } P = 0, \qquad \text{b) } y_i\psi^i_{kh} = 0.$$

In view of (3.6) and (2.2a), the condition (3.7a) leads to $H^i_{jkh} = 0$. Transvecting (3.1b) by y_i and using (3.7b) we have (3.2), and hence the manifold considered is flat. Thus, each of the conditions (3.7a) and (3.7b) leads to a flat manifold, which is a contradiction to our hypothesis.

From the above Theorems we may conclude that there does not exist any non-flat Finsler manifold whose curvature tensor is decomposable in any of the forms (3.1a) and (3.1b). Therefore, the possible decomposition of curvature tensor of a non-flat Finsler manifold is only (3.1c). Hence forward whenever we say that the curvature tensor is decomposable we shall mean its form (3.1c).

Now we shall discuss some non-flat Finsler manifolds whose curvature tensor is not decomposable. In this connection we propose:

Theorem 3.3. *The curvature tensor of a non-flat Finsler manifold with vanishing H is not decomposable.*

PROOF. Let us consider a non-flat Finsler manifold with vanishing H. If its curvature tensor is decomposable, we have (3.1c). Transvecting (3.1c) by $\dot{x}^j$ and using (2.1a), we have

$$H^i_{kh} = T^i Y_{kh}, \tag{3.8}$$

where $T^i = T^i_j \dot{x}^j$. Transvecting (3.8) by $\dot{x}^k$ and using (2.1b), we have

$$H^i_h = T^i Y_h \tag{3.9}$$

where $Y_h = Y_{kh}\dot{x}^k$. Transvecting (3.9) by g_{ik} we get

$$H_{h\cdot\cdot k} = T_k Y_h \tag{3.10}$$

where $T_k = g_{ik} T^i$. From (2.9) we know that the tensor $H_{k\cdot\cdot h}$ is symmetric, therefore from (3.10) we conclude

$$T_k Y_h = T_h Y_k, \tag{3.11}$$

which implies

$$Y_k = \varphi T_k, \tag{3.12}$$

where φ is some non-zero scalar. From (3.9) and (3.12) we get

$$H^i_h = \varphi T^i T_h. \tag{3.13}$$

Contracting i and h in (3.13) and using (2.1e), we have

$$(n-1)H = \varphi T^i T_i. \tag{3.14}$$

By hypothesis, $H=0$. Therefore, (3.14) gives $T^i T_i = 0$; which implies $T^i = 0$, since our metric is positive definite. From (3.8) and $T^i = 0$, we get $H^i_{kh} = 0$; which, in view of (2.2a), implies $H^i_{jkh} = 0$. Thus we see that the decomposability of curvature tensor leads to a contradiction.

THEOREM 3.4. *The curvature tensor of an isotropic Finsler manifold F_n $(n>2)$ is not decomposable.*

PROOF. Let us consider an isotropic Finsler manifold F_n $(n>2)$ characterized by

$$H^i_j = H(\delta^i_j - l^i l_j), \tag{3.15}$$

where $l^i = \dfrac{\dot{x}^i}{F}$. If its curvature tensor is decomposable, the tensor H^i_j satisfies (cf. (3.13))

$$H^i_j = \varphi T^i T_j. \tag{3.16}$$

The magnitude of vector T^i is given by

$$(n-1)H = \varphi T^i T_i. \tag{3.17}$$

Transvecting (3.16) by l^j and using $H^i_j l^j = 0$, we get $T_j l^j = 0$; which is equivalent to

$$T^j l_j = 0. \tag{3.18}$$

From (3.15) and (3.16) we get

$$\varphi T^i T_j = H(\delta^i_j - l^i l_j). \tag{3.19}$$

Transvecting (3.19) by T^j, and using (3.17) and (3.18) we have

$$(n-2)HT^i = 0. \tag{3.20}$$

Since $n>2$, we have either $H=0$ or $T^i=0$. In both cases $H^i_j=0$; which, in view of 2.2a) and (2.2b), implies $H^i_{jkh}=0$. Thus, we get a contradiction.

4. Necessary and sufficient condition for the decomposability of curvature tensor

In this section, we shall discuss a condition which is necessary and sufficient for the decomposability of curvature tensor of a Finsler manifold. For this purpose, we establish:

LEMMA 4.1. *Every tensor $T^{\dots}_{\dots}$ of type (p, q) which is skew-symmetric in its k^{th} and h^{th} lower indices is decomposable in the form*

$$T^{\dots}_{\dots} = A_{j_k j_h} \overset{*}{T}{}^{\dots}_{\dots} \tag{4.1}$$

where $\overset{*}{T}{}^{\dots}_{\dots}$ is some tensor of type $(p, q-2)$, if there exists a skew-symmetric tensor A_{lm} satisfying

$$A_{lm} T^{\dots}_{\dots j_k \dots j_h \dots} + A_{l j_h} T^{\dots}_{\dots m \dots j_k \dots} + A_{l j_k} T^{\dots}_{\dots j_h \dots m \dots} = 0. \tag{4.2}$$

$T^{\dots}_{\dots l \dots m \dots}$ means the k^{th} and h^{th} indices of the tensor $T^{\dots}_{\dots}$ are l and m, respectively.

PROOF. Let us consider a tensor $T^{\dots}_{\dots}$ satisfying the conditions of the above Lemma. By cyclic permutation of indices l, m, j_k and j_h in (4.2) we may write

$$A_{j_h l} T^{\dots}_{\dots m \dots j_k \dots} + A_{j_h j_k} T^{\dots}_{\dots l \dots m \dots} + A_{j_h m} T^{\dots}_{\dots j_k \dots l \dots} = 0, \tag{4.3}$$

$$A_{j_k j_h} T^{\dots}_{\dots l \dots m \dots} + A_{j_k m} T^{\dots}_{\dots j_h \dots l \dots} + A_{j_k l} T^{\dots}_{\dots m \dots j_h \dots} = 0, \tag{4.4}$$

and

$$A_{m j_k} T^{\dots}_{\dots j_h \dots l \dots} + A_{ml} T^{\dots}_{\dots j_k \dots j_h \dots} + A_{m j_h} T^{\dots}_{\dots l \dots j_k \dots} = 0. \tag{4.5}$$

Adding (4.2), (4.3), (4.4) and (4.5), and using the skew-symmetric properties of the tensors A_{lm} and $T^{\dots}_{\dots}$ we get

$$A_{l j_k} T^{\dots}_{\dots m \dots j_h \dots} = A_{m j_h} T^{\dots}_{\dots l \dots j_k \dots}; \tag{4.6}$$

which implies

$$T^{\dots}_{\dots m \dots j_h \dots} = A_{m j_h} \overset{*}{T}{}^{\dots}_{\dots}$$

where $\overset{*}{T}{}^{\dots}_{\dots}$ is some tensor of type $(p, q-2)$.

Utilizing this Lemma, we shall prove:

THEOREM 4.1. *The curvature tensor of a Finsler manifold is decomposable if and only if there exists a skew-symmetric tensor A_{kh} satisfying*

$$A_{lm} H^i_{jkh} + A_{lh} H^i_{jmk} + A_{lk} H^i_{jhm} = 0. \tag{4.7}$$

PROOF. If there exists a tensor A_{kh} satisfying (4.7), in view of Lemma (4.1), the curvature tensor H^i_{jkh} satisfies

$$H^i_{jkh} = B^i_j A_{kh}, \tag{4.8}$$

where B^i_j is some non-null tensor; and hence the curvature tensor is decomposable. Conversely, if the curvature tensor of a Finsler manifold is decomposable, it has the form (4.8). Since H^i_{jkh} is skew-symmetric in k and h, the tensor A_{kh} is skew-symmetric in k and h. Since the manifold is non-flat, in view of Lemma (3.1), $y_i H^i_{jkh} \neq 0$. Hence

$$y_i H^i_{jkh} = \lambda_j A_{kh} \tag{4.9}$$

where $\lambda_j = y_i B^i_j$. Transvecting the Bianchi identity

$$H^i_{jkh} + H^i_{hjk} + H^i_{khj} = 0 \tag{4.10}$$

by y_i and using (4.9), we have

$$\lambda_j A_{kh} + \lambda_h A_{jk} + \lambda_k A_{hj} = 0. \tag{4.11}$$

Since $\lambda_j \neq 0$, we may choose a vector v^i such that $v^j \lambda_j = 1$. Transvecting (4.11) by such vector v^j we get

$$A_{kh} = \lambda_k \mu_h - \lambda_h \mu_k \tag{4.12}$$

where $\mu_h = A_{jh} v^j$. Obviously μ_k satisfies

$$\mu_j A_{kh} + \mu_h A_{jk} + \mu_k A_{hj} = 0. \tag{4.13}$$

Transvecting (4.13) and (4.11) by λ_i and μ_i respectively, and then subtracting we have

$$A_{ij} A_{kh} + A_{ih} A_{jk} + A_{ik} A_{hj} = 0. \tag{4.14}$$

Transvecting (4.8) by A_{lm}, taking skew-symmetric part with respect to the indices m, k and h, and using (4.14) we get (4.7).

REMARK 4.1. In case of Riemannian and non-Riemannian manifolds, it is not yet established that the possible decomposition of curvature tensor is only of the form (4.8). Therefore, in these cases the corresponding theorem is given by "*The curvature tensor R^i_{jkh} of a Riemannian or non-Riemannian manifold satisfies $R^i_{jkh} = B^i_j A_{kh}$ if there exists a skew symmetric tensor A_{kh} satisfynig*

$$A_{lm} R^i_{jkh} + A_{lh} R^i_{jmk} + A_{lk} R^i_{jhm} = 0.\text{"} \tag{4.15}$$

REMARK 4.2. In case of a Riemannian manifold, if the curvature tensor R^i_{jkh} satisfies $R^i_{jkh} = B^i_j A_{kh}$, the associate curvature tensor $R_{ijkh} \overset{\text{def}}{=\!=} g_{il} R^l_{jkh}$ satisfies

$$R_{ijkh} = B_{ij} A_{kh} \tag{4.16}$$

where $B_{ij} = g_{il} B^l_j$. In view of $R_{ijkh} = R_{khij}$, (4.16) gives

$$B_{ij} = \varphi A_{ij}, \tag{4.17}$$

where φ is some non-zero scalar. From (4.16) and (4.17) we have $R_{ijkh} = \varphi A_{ij} A_{kh}$; which may be written as

$$R_{ijkh} = \bar{A}_{ij} \bar{A}_{kh}, \tag{4.18}$$

where $\bar{A}_{ij} = \sqrt{\varphi} A_{ij}$. Conversely, if R_{ijkh} satisfies (4.18), transvection of (4.18) by g^{il} immediately gives $R^i_{jkh} = \bar{A}^i_j \bar{A}_{kh}$. Therefore we have "*The associate curvature tensor of a Riemannian manifold satisfies* (4.18) *if and only if there exists a skew-symmetric tensor $\bar{A}_{lm}$ satisfying*

$$\bar{A}_{lm} R_{ijkh} + \bar{A}_{lh} R_{ijmk} + \bar{A}_{lk} R_{ijhm} = 0.\text{"} \tag{4.19}$$

REMARK 4.3. In case of birecurrent Riemannian (non-Riemannian) manifolds the curvature tensor R^i_{jkh} satisfies [4]

$$T_{lm} R^i_{jkh} + T_{lh} R^i_{jmk} + T_{lk} R^i_{jhm} = 0,$$

where T_{lm} is the tensor of recurrence. If T_{lm} is skew-symmetric, in view of Remark (4.1), we have $R^i_{jkh} = B^i_j T_{kh}$.

REMARK 4.4. The recurrence tensor a_{lm} of a birecurrent Finsler manifold satisfies ([3], [5])

$$a_{lm} H^i_{kh} + a_{lh} H^i_{mk} + a_{lk} H^i_{hm} = 0.$$

If a_{lm} is skew-symmetric, in view of Lemma (4.1), we have

$$H^i_{kh} = X^i a_{kh}. \tag{4.20}$$

Differentiating (4.20) partially with respect to $\dot{x}^j$ and using (2.2a) we have

$$H^i_{jkh} = (\dot{\partial}_j X^i) a_{kh} + X^i \dot{\partial}_j a_{kh}.$$

If the tensor a_{kh} is independent of $\dot{x}^i$'s, we have $H^i_{jkh} = (\dot{\partial}_j X^i) a_{kh}$. Thus we conclude, "*The curvature tensor of a birecurrent Finsler manifold is decomposable if the recurrence tensor is skew-symmetric and independent of $\dot{x}^i$'s.*

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ESTIMATE FOR THE DIFFERENCE OF THE SECOND DERIVATIVE OF PARTIAL SUMS ARISING BY NON-NEGATIVE SELFADJOINT EXTENSIONS OF STURM—LIOUVILLE OPERATORS

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Introduction

In this paper we prove exact (concerning the order) estimates for the difference of the second derivatives of the partial sums of expansions, one of which corresponds to the non-negative selfadjoint extension of the operator $Lu=-u''+q(x)u(x)$ $\big(u\in C_0^\infty(G)$, G is a finite open interval$\big)$ defined by the boundary condition $u'(a)=$ $=u'(b)=0$, and the other corresponds to the non-negative selfadjoint extension of the operator $\hat{L}u=-u''+\hat{q}(x)u(x)$ $\big(u\in C_0^\infty(G)$, G is the same as before$\big)$ defined by the same boundary conditions as above, i.e. $u'(a)=u'(b)=0$. We assume that q and $\hat{q}$ alow such extensions; further q and $\hat{q}$ belong to the class $C^{(\alpha)}(G)\cap L_p(G)$ $(0<\alpha<1,$ $1<p<\infty)$.

Assuming that f has a compact support in G we obtain the equiconvergence (uniform on every compact subset of G) of the second derivative of the partial sums defined above. If we take $\hat{L}u=-u''$ (i.e. $\hat{q}\equiv 0$) then the problem of convergence of the second derivative of the partial sums of an absolutely continuous function with compact support, with respect to the eigenfunction of a Sturm—Liouville operator (in the case of the second boundary value problem), can be reduced to the trigonometric case.

Now we formulate our results. Let $G=(a, b)$ be a finite interval on the real line and consider non-negative selfadjoint extensions of the symmetric operators

$$Lu=-u''+q(x)u(x),\quad \hat{L}u=-u''+q(x)u(x),\quad u\in C_0^\infty(G)$$

defined by the second boundary condition $u'(a)=u'(b)=0$. Suppose

$$q \text{ and } \hat{q}\in C^{(\alpha)}(G)\cap L_p(G)\quad (0<\alpha<1,\ 1<p<\infty). \tag{1}$$

Denote $\{u_n\}$ and $\{\hat{u}_n\}$ the complete orthonormal systems of the corresponding extensions; further let $\{\lambda_n\}$ and $\{\hat{\lambda}_n\}$ be the eigenvalues $(\lambda_n\geqq 0;\ \hat{\lambda}_n\geqq 0;\ \lambda_1\leqq\lambda_2\leqq\ldots;$ $\hat{\lambda}_1\leqq\hat{\lambda}_2\leqq\ldots)$. Let $f(x)$ be a function with absolutely continuous derivative on the closed interval $[a, b]$; further let μ be an arbitrary non-negative number. Consider the partial sums

$$\sigma_\mu(f, x)=\sum_{\sqrt{\lambda_n}<\mu}{}_n (f, u_n)\cdot u_n(x),\quad \hat{\sigma}_\mu(f, x)=\sum_{\sqrt{\hat{\lambda}_n}<\mu}{}_n (f, \hat{u}_n)\cdot \hat{u}_n(x) \tag{2}$$

of the expansions of f with respect to the systems $\{u_n\}$ and $\{\hat{u}_n\}$, respectively. We shall prove the following theorems.

Theorem 1. *Suppose the function f is such that its derivative is absolutely continuous on the closed interval $[a, b]$ and for the potentials the condition* (1) *is fulfilled.*

Then the estimate

$$\frac{d^2}{dx^2}\sigma_\mu(f,x)-\frac{d^2}{dx^2}\hat{\sigma}_\mu(f,x)=O(1)\quad(\mu\geqq 0)\tag{3}$$

holds uniformly on every compact subset of the open interval $G=(a,b)$.

THEOREM 2. *If the conditions of Theorem 1 are fulfilled, further f has a compact support in G and q'', $\hat{q}''\in L_1(G)$, then instead of* (3) *the stronger estimate*

$$\frac{d^2}{dx^2}\sigma_\mu(f,x)-\frac{d^2}{dx^2}\hat{\sigma}_\mu(f,x)=o(1)\quad(\mu\to\infty)\tag{4}$$

holds, uniformly on every compact subset of G. The order of the estimates (3) *and* (4) *cannot be improved.*

The proof of these theorems is based on some results obtained in [2] and [4] hence we formulate them below (further we shall use some ideas and methods developed in [5]):

If G is a finite interval and $q\in L_1(G)$ then for an arbitrary fundamental system of the operator $Lv=-v''+q(x)v(x)$ the estimate (cf. [2])

$$\sup_{x\in G}|v_n(x)|\leqq C\quad(n=1,2,\ldots)\tag{5}$$

holds. If $q\in L_p(G)$ and $1<p$, then we have (cf. [2])

$$\sum_{|\sqrt{\lambda_n}-\mu|\leqq 1}1\leqq D\quad(\mu\geqq 1).\tag{6}$$

If G is a finite or infinite interval and $q\in C(G)$, then the estimate

$$\max_{x\in K}|v_n''(x)|\leqq C(K)\big[(\sqrt{\lambda_n})^2+1\big]\quad(n=1,2,\ldots)\tag{7}$$

holds true uniformly on every compact subset K of G (cf. [4]). Here the constants C and $C(K)$ do not depend on n and D does not depend on μ and n, further $\{\lambda_n\}$ denotes the system of the corresponding fundamental numbers. (For the terminology see [2].)

This paper consists of four sections. In §1 we give an estimate for the derivative and for the integral of the eigenfunctions of the operators considered. These estimates play an important role in the proof and they are interesting in themselves. In §2 we estimate the second derivative of the spectral function of the operator considered and in § 3 we prove Theorems 1 and 2. At last in §4 we consider arbitrary non-negative selfadjoint extensions of the operators L and $\hat{L}$. We give the modifications necessary to obtain results in this general case.

§1. Estimate for the derivative and for the integral of the eigenfunctions

LEMMA 1. *Suppose $q\in C(G)\cap L_p(G)$ $(1<p)$. Then for an arbitrary fundamental system $\{v_n\}$ of the operator L the estimate*

$$\sum_{|\sqrt{\lambda_n}-\mu|\leqq 1}\left|\frac{d^2}{dx^2}v_n(x)\right|^2\leqq C(K)\mu^4\quad(\mu\geqq 1)\tag{8}$$

holds uniformly in x on every compact subset K of G and the constant $C(K)$ depends only on K and on q.

PROOF. The estimate (8) follows from (6) and (7).

LEMMA. 2. *Suppose $q\in L_p(G)$ $(1<p)$, $|b|<\infty$ and let for the fundamental functions the condition $v_n'(a)=v_n'(b)=0$ be fulfilled. Then the estimate*

$$\sum_{|\sqrt{\lambda_n}-\mu|\leqq 1}\left|\int_{y_1}^{y_2}\Big(\int_a^y v_n(\xi)\,d\xi\Big)dy\right|^2\leqq\frac{C}{\mu^4}\quad(\mu\geqq 1)\tag{9}$$

holds uniformly in y_1 and y_2 $(a\leqq y_1\leqq y_2\leqq b)$ and the constant C does not depend on n and μ.

PROOF. According to the definition of the fundamental function, we have

$$-v_n''(x)+q(x)v_n(x)=\lambda_n v_n(x)\tag{10}$$

for almost every $x\in G$. Let $\lambda_n\neq 0$. Then taking into consideration

$$v_n'(a)=v_n'(b)=0,\tag{11}$$

from (10) we obtain

$$\int_a^y v_n(\xi)\,d\xi=-\frac{1}{\lambda_n}[v_n'(y)-v_n'(a)]+\frac{1}{\lambda_n}\int_a^y q(\xi)v_n(\xi)\,d\xi=$$
$$=-\frac{1}{\lambda_n}v_n'(y)+\frac{1}{\lambda_n}\int_a^y q(\xi)v_n(\xi)\,d\xi.$$

Integrating with respect to y on the interval $y_1\leqq y\leqq y_2$ we obtain

$$\int_{y_1}^{y_2}\Big(\int_a^y v_n(\xi)\,d\xi\Big)dy=-\frac{1}{\lambda_n}[v_n(y_2)-v_n(y_1)]+\frac{1}{\lambda_n}\int_{y_1}^{y_2}\Big(\int_a^y q(\xi)v_n(\xi)\,d\xi\Big)dy.$$

Hence, using (5), the estimate

$$\left|\int_{y_1}^{y_2}\Big(\int_a^y v_n(\xi)\,d\xi\Big)dy\right|\leqq\frac{C}{\lambda_n},\quad\lambda_n\neq 0$$

follows, which gives (9) if we take into consideration (6). One can prove the estimate

$$\sum_{|\sqrt{\lambda_n}-\mu|\leqq 1}\left|\int_{y_1}^{y_2}\Big(\int_y^b v_n(\xi)\,d\xi\Big)dy\right|^2\leqq\frac{C}{\mu^4}\quad(\mu\geqq 1)\tag{9'}$$

by the same method.

§2. Estimate for the second derivative of the spectral function

Consider the non-negative selfadjoint extension of the operator L defined by the second boundary condition (11). Denote $\theta(x, y, \mu)$ the spectral function of this selfadjoint operator, i.e.

$$\theta(x, y, \mu) \overset{\text{def}}{=} \sum_{\sqrt{\lambda_n}<\mu} u_n(x)\, u_n(y). \tag{12}$$

Let K be an arbitrary compact subset of G and fix a number R_0 from the interval $(0, 1/2 \operatorname{dist}(K, \partial G))$ arbitrarily, further suppose $R_0 \leqq R \leqq 2R_0$. Denote S_{R_0} the "average operator" introduced by V. A. Il'in [1]:

$$S_R f = S_{R_0}[f] \overset{\text{def}}{=} \frac{1}{R_0} \int_{R_0}^{2R_0} f(R)\, dR.$$

Consider the (modified) trigonometric spectral function

$$v_R(x, y, \mu) = \begin{cases} \dfrac{1}{\pi} \cdot \dfrac{\sin \mu|x-y|}{|x-y|} & \text{if } |x-y| < R \\ 0 & \text{if } |x-y| \geqq R \end{cases} \qquad (x \in K,\ y \in G,\ \mu > 0). \tag{13}$$

In this section we shall prove the following

LEMMA 3. *Suppose* $q \in C^{(\alpha)}(G) \cap L_p(G)$ $(0<\alpha<1,\ 1<p<\infty)$. *Then the estimate*

$$\left| \int_{y_1}^{y_2} \left(\int_a^y \frac{\partial^2}{\partial x^2} \{S_{R_0}[v_R(x, \xi, \mu)] - \theta(x, \xi, \mu)\}\, d\xi \right) dy \right| \leqq D(K) \quad (\mu \geqq 1) \tag{14}$$

holds uniformly in x *on every compact subset* K *of* G *and in* y_1, y_2 *on the interval* $a \leqq y_1 \leqq y_2 \leqq b$. *The constant* $D(K)$ *depends only on* K *and* q.

PROOF. Fix a point x in K arbitrarily (we may suppose that K is a closed subinterval of G). Consider the function

$$\omega(x, y, \mu) = \begin{cases} \dfrac{1}{\pi} \dfrac{\sin \mu R}{R} & \text{if } |x-y| < R \\ 0 & \text{if } |x-y| \geqq R \end{cases} \qquad (x \in K,\ y \in G,\ \mu > 0).$$

In [3] it is proved (cf. [3], relation (42)) that for arbitrarily fixed $x \in K$ and $\mu \geqq 1$ we have

$$\int_a^y \{S_{R_0}[v_R(x, \xi, \mu)] - \theta(x, \xi, \mu)\}\, d\xi = \tag{15}$$

$$= \int_a^y S_{R_0}[\omega_R(x, \xi, \mu)]\, d\xi + \frac{1}{2} \sum_{\sqrt{\lambda_n}=\mu} \left(\int_a^y u_n(\xi)\, d\xi \right) u_n(x) -$$

$$-\frac{2}{\pi} \sum_{0 \leqq \sqrt{\lambda_n} \leqq 1} \left(\int_a^y u_n(\xi)\, d\xi \right) \cdot u_n(x) \cdot S_{R_0} \left[\frac{\sin \mu R \sin \sqrt{\lambda_n} R}{R \sqrt{\lambda_n}} \right] -$$

$$- \sum_{0 \leqq \sqrt{\lambda_n} \leqq 1} \left(\int_a^y u_n(\xi)\, d\xi \right) \cdot u_n(x) \cdot S_{R_0}[I^{\mu}_{\sqrt{\lambda_n}}(R)] -$$

$$- \sum_{\sqrt{\lambda_n}>1} \left(\int_a^y u_n(\xi)\, d\xi \right) u_n(x) \frac{\mu}{\sqrt{\lambda_n}} S_{R_0}[K^{\mu}_{\sqrt{\lambda_n}}(R)] -$$

$$-\frac{1}{\pi} \sum_{n=1}^{\infty} \left(\int_a^y u_n(\xi)\, d\xi \right) \cdot S_{R_0} \left[\int_0^R \frac{\sin \mu h}{h} \left[\int_{x-h}^{x+h} q(\xi)\, u_n(\xi) \frac{\sin \sqrt{\lambda_n}(|x-\xi|-h)}{\sqrt{\lambda_n}}\, d\xi \right] dh \right] +$$

$$+\frac{1}{\pi} \sum_{n=1}^{\infty} \left(\int_a^y u_n(\xi)\, d\xi \right) \cdot S_{R_0} \left[\frac{\sin \mu R}{R} \int_0^R \int_{x-h}^{x+h} q(\xi) u_n(\xi) \frac{\sin \sqrt{\lambda_n}(|x-\xi|-h)}{\sqrt{\lambda_n}}\, d\xi\, dh \right]$$

$$(a \leqq y \leqq b),$$

where

$$I^{\mu}_{\sqrt{\lambda_n}}(R) = \frac{2}{\pi} \int_R^{\infty} \frac{\sin \mu h \cos \sqrt{\lambda_n} h}{h}\, dh, \quad K^{\mu}_{\sqrt{\lambda_n}}(R) = -\frac{2}{\pi\mu} \int_R^{\infty} \frac{\partial}{\partial h} \left(\frac{\sin \mu h}{h} \right) \sin \sqrt{\lambda_n}\, h\, dh.$$

It is easy to prove by the method of [7] that all the series on the right hand side of (15) converge uniformly in y on the interval $[a, b]$. Hence, integrating in y on the interval $[y_1, y_2]$ $(a \leqq y_1 \leqq y_2 \leqq b)$, we obtain the following equality for arbitrary fixed $x \in K$ and $\mu \geqq 1$:

$$\int_{y_1}^{y_2} \left(\int_a^y [S_{R_0} v_R(x, \xi, \mu) - \theta(x, \xi, \mu)]\, d\xi \right) dy = \tag{16}$$

$$= \int_{y_1}^{y_2} \left(\int_a^y S_{R_0} \omega_R(x, \xi, \mu)\, d\xi \right) dy + \frac{1}{2} \sum_{\sqrt{\lambda_n}=\mu} \left[\int_{y_1}^{y_2} \left(\int_a^y u_n(\xi)\, d\xi \right) dy \right] \cdot u_n(x) -$$

$$-\frac{2}{\pi} \sum_{0 \leqq \sqrt{\lambda_n} \leqq 1} \left[\int_{y_1}^{y_2} \left(\int_a^y u_n(\xi)\, d\xi \right) dy \right] \cdot u_n(x) \cdot S_{R_0} \left[\frac{\sin \mu R \sin \sqrt{\lambda_n} R}{R \sqrt{\lambda_n}} \right] -$$

$$- \sum_{0 \leqq \sqrt{\lambda_n} \leqq 1} \left[\int_{y_1}^{y_2} \left(\int_a^y u_n(\xi)\, d\xi \right) dy \right] \cdot u_n(x) \cdot S_{R_0} I^{\mu}_{\sqrt{\lambda_n}}(R) - \sum_{\sqrt{\lambda_n}>1} \left[\int_{y_1}^{y_2} \left(\int_a^y u_n(\xi)\, d\xi \right) dy \right] \cdot$$

$$\cdot u_n(x) \cdot \frac{\mu}{\sqrt{\lambda_n}} \cdot S_{R_0} K^{\mu}_{\sqrt{\lambda_n}}(R) - \frac{1}{\pi} \sum_{n=1}^{\infty} \left[\int_{y_1}^{y_2} \left(\int_a^y u_n(\xi)\, d\xi \right) dy \right] \cdot$$

$$\cdot S_{R_0} \left[\int_0^R \frac{\sin \mu h}{h} \left[\int_{x-h}^{x+h} q(\xi)\, u_n(\xi) \frac{\sin \sqrt{\lambda_n}(|x-\xi|-h)}{\sqrt{\lambda_n}}\, d\xi \right] dh \right] +$$

$$+\frac{1}{\pi} \sum_{n=1}^{\infty} \left[\int_{y_1}^{y_2} \left(\int_a^y u_n(\xi)\, d\xi \right) dy \right] \cdot$$

$$\cdot S_{R_0} \left[\frac{\sin \mu R}{R} \int_0^R \left[\int_{x-h}^{x+h} q(\xi)\, u_n(\xi) \frac{\sin \sqrt{\lambda_n}(|x-\xi|-h)}{\sqrt{\lambda_n}}\, d\xi \right] dh \right].$$

Denote $g(x)$ $(h(x))$ the left (right) hand side of (16). Our aim is to prove: the derivatives $(d^2/dx^2)g(x)$, $(d^2/dx^2)h(x)$ exist and are equal.

It is easy to see that

$$S_{R_0}[v_R(x,y,\mu)]=\begin{cases}\dfrac{1}{\pi}\cdot\dfrac{\sin\mu|x-y|}{|x-y|} & \text{if } |x-y|\leqq R_0\\[2mm] \dfrac{1}{\pi}\cdot\dfrac{\sin\mu|x-y|}{|x-y|}\cdot\dfrac{2R_0-|x-y|}{R_0} & \text{if } R_0\leqq|x-y|\leqq 2R_0\\[2mm] 0 & \text{if } |x-y|>2R_0.\end{cases}$$

Using the notation

(17)

$$\frac{\partial^2}{\partial x^2}S_{R_0}v_R(x,y,\mu)=\begin{cases}\dfrac{1}{\pi}\cdot\dfrac{\partial^2}{\partial x^2}\dfrac{\sin\mu|x-y|}{|x-y|} & \text{if } |x-y|\leqq R_0,\\[2mm] \dfrac{1}{\pi}\dfrac{\partial^2}{\partial x^2}\left(\dfrac{\sin\mu|x-y|}{|x-y|}\cdot\dfrac{2R_0-|x-y|}{R_0}\right) & \text{if } R_0\leqq|x-y|\leqq 2R_0,\\[2mm] 0 & \text{if } |x-y|>2R_0\end{cases}$$

we have[1]

(18)

$$\frac{d^2}{dx^2}g(x)=\int_{y_1}^{y_2}\left(\int_a^y\frac{\partial^2}{\partial x^2}[S_{R_0}v_R(x,\xi,\mu)-\theta(x,\xi,\mu)]\,d\xi\right)dy+F(x,y_1,y_2,\mu,R_0),$$

where $F(x,y_1,y_2,\mu,R_0)$ is such a function for which the estimate

$$|F|\leqq C(K)\quad(x\in K,\ \mu\geqq 0,\ a\leqq y_1\leqq y_2\leqq b) \tag{19}$$

holds uniformly in x on the compact set K. The constant $C(K)$ depends only on K.

Denote $h_i(x)$ $(i=,\dots,7)$ the ith member on the right hand side of (16). It is easy to see by the method of [3], that the derivatives $(\partial^2/\partial x^2)h_i(x)$ $(i=2,3,4)$ exist. Now we prove: $(\partial^2/\partial x^2)h_1(x)$ exists; further

$$\frac{d^2}{dx^2}\int_{y_1}^{y_2}\left(\int_a^y S_{R_0}\omega_R(x,\xi,\mu)\,d\xi\right)dy= \tag{20}$$

$$=\int_{y_1}^{y_2}\left(\int_a^y\frac{d^2}{dx^2}S_{R_0}\omega_R(x,\xi,\mu)\,d\xi\right)dy+F_1(x,y_1,y_2,\mu,R_0)$$

where for the function F_1 the estimate

$$|F_1|\leqq C(K)\quad(x\in K,\ \mu\geqq 1,\ a\leqq y_1\leqq y_2\leqq b) \tag{21}$$

---

[1] The proof of (18) is similar to that of (20) but it is very tedious (there are many cases which should be distinguished), so we omit the details.

holds. To this end we start from the obvious relation

$$S_{R_0}\omega_R(x,\xi,\mu)=\begin{cases}0 & \text{if } x-2R_0\leqq\xi,\\[1mm] \dfrac{1}{R_0}\displaystyle\int_{x-\xi}^{2R_0}\dfrac{\sin\mu R}{R}\,dR & \text{if } x-2R_0\leqq\xi\leqq x-R_0,\\[2mm] \dfrac{1}{R_0}\displaystyle\int_{R_0}^{2R_0}\dfrac{\sin\mu R}{R}\,dR & \text{if } x-R_0\leqq\xi\leqq x+R_0,\\[2mm] \dfrac{1}{R_0}\displaystyle\int_{\xi-x}^{2R_0}\dfrac{\sin\mu R}{R}\,dR & \text{if } x+R_0\leqq\xi\leqq x+2R_0,\\[1mm] 0 & \text{if } x+2R_0\leqq\xi.\end{cases}$$

Using the notation $[c,d]=K$, consider the case $y_1\in(a,c-2R_0)$, $y_2\in(c-R_0,R_0]$, $x\in[c,y_2+R_0)$. (The other cases are verified analogously.) An easy calculation shows that

$$\int_{y_1}^{y_2}\left(\int_a^y S_{R_0}\omega_R(x,\xi,\mu)\,d\xi\right)dy=\int_{x-2R_0}^{x-R_0}\int_a^y+\int_{x-R_0}^{y_2}\int_a^y=$$

$$=\int_{x-2R_0}^{x-R_0}\left(\int_{x-2R_0}^{y}\left(\frac{1}{R_0}\int_{x-\xi}^{2R_0}\frac{\sin\mu R}{R}\,dR\right)d\xi\right)dy+$$

$$+\int_{x-R_0}^{y_2}\left(\int_{x-2R_0}^{x-R_0}\left(\frac{1}{R_0}\int_{x-\xi}^{2R_0}\frac{\sin\mu R\,dR}{R}\right)d\xi\right)dy+$$

$$+\int_{x-R_0}^{y_2}\left(\int_{x-R_0}^{y}\left(\frac{1}{R_0}\int_{R_0}^{2R_0}\frac{\sin\mu R}{R}\,dR\right)d\xi\right)dy$$

and

$$\frac{d}{dx}\int_{y_1}^{y_2}\left(\int_a^y S_{R_0}\omega_R(x,\xi,\mu)\,d\xi\right)dy=\int_{x-2R_0}^{x-R_0}\left(\frac{d}{dx}\left(\int_{x-2R_0}^{y}\left(\frac{1}{R_0}\int_{x-\xi}^{2R_0}\frac{\sin\mu R\,dR}{R}\right)d\xi\right)\right)dy+$$

$$+\int_{x-2R_0}^{x-R_0}\left(\frac{1}{R_0}\int_{x-\xi}^{2R_0}\frac{\sin\mu R}{R}\,dR\right)d\xi+\int_{x-R_0}^{y_2}\left(\frac{d}{dx}\left(\int_{x-2R_0}^{x-R_0}\left(\frac{1}{R_0}\int_{x-\xi}^{2R_0}\frac{\sin\mu R}{R}\,dR\right)d\xi\right)\right)dy-$$

$$-\int_{x-2R_0}^{x-R_0}\left(\frac{1}{R_0}\int_{x-\xi}^{2R_0}\frac{\sin\mu R}{R}\,dR\right)d\xi+\int_{x-R_0}^{y_2}\left(\frac{d}{dx}\left(\int_{x-R_0}^{y}\left(\frac{1}{R_0}\int_{R_0}^{2R_0}\frac{\sin\mu R}{R}\,dR\right)d\xi\right)\right)dy-0=$$

$$=\int_{x-2R_0}^{x-R_0}\left(\int_{x-2R_0}^{y}\frac{d}{dx}\left(\frac{1}{R_0}\int_{x-\xi}^{2R_0}\frac{\sin\mu R}{R}\,dR\right)d\xi\right)dy+\int_{x-2R_0}^{x-R_0}\left(-\frac{1}{R_0}\int_{2R_0}^{2R_0}\frac{\sin\mu R}{R}\,dR\right)dy+$$

$$+\int_{x-R_0}^{y_2}\left(\int_{x-2R_0}^{x-R_0}\frac{d}{dx}\left(\frac{1}{R_0}\int_{x-\xi}^{2R_0}\frac{\sin\mu R}{R}\,dR\right)d\xi\right)dy+\int_{x-R_0}^{y_2}\left(\frac{1}{R_0}\int_{R_0}^{2R_0}\frac{\sin\mu R}{R}\,dR\right)dy-0+$$

$$+\int_{x-2R_0}^{y_2}\left(\int_{x-2R_0}^{y}\frac{d}{dx}\left(\frac{1}{R_0}\int_{R_0}^{2R_0}\frac{\sin\mu R}{R}\,dR\right)d\xi\right)dy-\int_{x-2R_0}^{y_2}\left(\frac{1}{R_0}\int_{R_0}^{2R_0}\frac{\sin\mu R}{R}\,dR\right)dy.$$

Hence

$$\frac{d^2}{dx^2}\int_{y_1}^{y_2}\Big(\int_a^y S_{R_0}\omega_R(x,\xi,\mu)\,d\xi\Big)dy=$$

$$=\int_{x-2R_0}^{x-R_0}\left(\frac{d}{dx}\left(\int_{x-2R_0}^{y}\frac{d}{dx}\left(\frac{1}{R_0}\int_{x-\xi}^{2R_0}\frac{\sin\mu R}{R}\,dR\right)d\xi\right)\right)dy+$$

$$+\int_{x-2R_0}^{x-R_0}\frac{d}{dx}\left(\frac{1}{R_0}\int_{x-\xi}^{2R_0}\frac{\sin\mu R}{R}\,dR\right)d\xi+$$

$$+\int_{x-R_0}^{y_2}\left(\frac{d}{dx}\left(\int_{x-2R_0}^{x-R_0}\frac{d}{dx}\left(\frac{1}{R_0}\int_{x-\xi}^{2R_0}\frac{\sin\mu R}{R}\,dR\right)d\xi\right)\right)dy-$$

$$-\int_{x-2R_0}^{x-R_0}\frac{d}{dx}\left(\frac{1}{R_0}\int_{x-\xi}^{2R_0}\frac{\sin\mu R}{R}\,dR\right)d\xi+\int_{x-2R_0}^{y_2}\Big(\int_{x-2R_0}^{y}0\,d\xi\Big)dy=$$

$$=\int_{x-2R_0}^{x-R_0}\left(\int_{x-2R_0}^{y}\frac{d^2}{dx^2}\left(\frac{1}{R_0}\int_{x-\xi}^{2R_0}\frac{\sin\mu R}{R}\,dR\right)d\xi\right)dy+$$

$$+\int_{x-2R_0}^{x-R_0}\left[-\frac{d}{dx}\left(\frac{1}{R_0}\int_{x-\xi}^{2R_0}\frac{\sin\mu R}{R}\,dR\right)\Big|_{\xi=x-2R_0}\right]dy+$$

$$+\int_{x-R_0}^{y_2}\left(\int_{x-2R_0}^{x-R_0}\frac{d^2}{dx^2}\left(\frac{1}{R_0}\int_{x-\xi}^{R_0}\frac{\sin\mu R}{R}\,dR\right)d\xi\right)dy+$$

$$+\int_{x-R_0}^{y_2}\left[\frac{d}{dx}\left(\frac{1}{R_0}\int_{x-\xi}^{2R_0}\frac{\sin\mu R}{R}\,dR\right)\Big|_{\xi=x-2R_0}^{\xi=x-R_0}\right]dy+\int_{x-2R_0}^{y_2}\Big(\int_{x-R_0}^{y}0\,d\xi\Big)dy=$$

$$=\int_{x-2R_0}^{x-R_0}\left(\int_{x-2R_0}^{y}\frac{d^2}{dx^2}\left(\frac{1}{R_0}\int_{x-\xi}^{2R_0}\frac{\sin\mu R}{R}\,dR\right)d\xi\right)dy+\int_{x-2R_0}^{x-R_0}\frac{\sin\mu R_0}{R_0}\,dy+$$

$$+\int_{x-R_0}^{y_2}\left(\int_{x-2R_0}^{x-R_0}\frac{d^2}{dx^2}\left(\frac{1}{R_0}\int_{x-\xi}^{2R_0}\frac{\sin\mu R}{R}\,dR\right)d\xi\right)dy+\int_{x-R_0}^{y_2}\left(\frac{\sin\mu\, 2R_0}{2R_0^2}-\frac{\sin\mu R_0}{R_0^2}\right)dy+0,$$

and the relations (20) and (21) follow.

Now consider the function $h_5(x)$. For the quantity $S_{R_0}[K^{\mu}_{\sqrt{\lambda_n}}(R)]$ the estimate

$$|S_{R_0}K^{\mu}_{\sqrt{\lambda_n}}(R)|\leqq C_1(R_0) \tag{22}$$

holds (cf. [1], (3.61)).

Using (6), (7), ((9) of [3]), (22) we obtain the estimates

$$\sum_{\sqrt{\lambda_n}>1}\left|\int_{y_1}^{y_2}\Big(\int_a^y u_n(\xi)\,d\xi\Big)dy\right|\cdot\left|\frac{d}{dx}u_n(x)\right|\cdot\frac{\mu}{\sqrt{\lambda_n}}\cdot|S_{R_0}K^{\mu}_{\sqrt{\lambda_n}}(R)|\leqq$$

$$\leqq C(K,R_0)\cdot\mu\sum_{\sqrt{\lambda_n}>1}\frac{1}{\lambda_n}=C(K,R_0)\cdot\mu\cdot\sum_{k=1}^{\infty}\Big(\sum_{k=\sqrt{\lambda_n}\leqq k+1}\frac{1}{\lambda_n}\Big)\leqq$$

$$\leqq C(K,R_0)\cdot\mu\cdot\sum_{k=1}^{\infty}\frac{1}{K^2}<\infty.$$

From this it follows that the derivative $\frac{d}{dx}h_5(x)$ exists and

$$\frac{d}{dx}h_5(x)=\sum_{\sqrt{\lambda_n}>1}\Big[\int_{y_1}^{y_2}\Big(\int_a^y u_n(\xi)\,d\xi\Big)dy\Big]\cdot u_n'(x)\cdot\frac{\mu}{\sqrt{\lambda_n}}\cdot S_{R_0}K^{\mu}_{\sqrt{\lambda_n}}(R).$$

One can prove the uniform convergence of the series (in x on K)

$$\sum_{\sqrt{\lambda_n}>1}\Big[\int_{y_1}^{y_2}\Big(\int_a^y u_n(\xi)\,d\xi\Big)dy\Big]\cdot\frac{d^2}{dx^2}u_n(x)\cdot\frac{\mu}{\sqrt{\lambda_n}}\cdot S_{R_0}K^{\mu}_{\sqrt{\lambda_n}}(R)=S \tag{23}$$

by the method used in [3] for the proof of the uniform convergence of the series (45). This proves the existence of the derivative $h_5''(x)$ and

$$\frac{d^2}{dx^2}h_5(x)=S.$$

Now consider the function

$$h_6(x)=\frac{1}{\pi}\sum_{n=1}^{\infty}\Big[\int_{y_1}^{y_2}\Big(\int_a^y u_n(\xi)\,d\xi\Big)dy\Big]\cdot \tag{24}$$

$$\cdot S_{R_0}\int_0^R\frac{\sin\mu h}{h}\int_{x-h}^{x+h}q(\xi)\,u_n(\xi)\frac{\sin\sqrt{\lambda_n}(|x-\xi|-h)}{\sqrt{\lambda_n}}\,d\xi\,dh.$$

The function

$$I_n(x,R)\overset{\text{def}}{=}\int_0^R\frac{\sin\mu h}{h}\int_{x-h}^{x+h}q(\xi)\,u_n(\xi)\frac{\sin\sqrt{\lambda_n}(|x-\xi|-h)}{\sqrt{\lambda_n}}\,d\xi\,dh,$$

and also $\frac{\partial}{\partial x}I_n(x,R)$ are continuous in (x,R) on the set $K\times[R_0,2R_0]$, hence

$$\frac{d}{dx}S_{R_0}I_n(x,R)=S_{R_0}\Big(\frac{\partial}{\partial x}I_n(x,R)\Big)$$

follows. From this it follows that for the proof of the existence of $\frac{d}{dx}h_0(x)$ it is enough to prove the uniform convergence in x (on K) of the series

$$\sum_{n=1}^{\infty}\Big[\int_{y_1}^{y_2}\Big(\int_a^y u_n(\xi)\,d\xi\Big)dy\Big]\cdot S_{R_0}\Big(\frac{\partial}{\partial x}I_n(x,R)\Big).$$

This fact follows from the estimates

$$\sum_{n=1}^{\infty}\Big|\int_{y_1}^{y_2}\Big(\int_a^y u_n(\xi)\,d\xi\Big)dy\Big|\cdot\Big|S_{R_0}\int_0^R\frac{\sin\mu h}{h}\Big[\int_{x-h}^{x}q(\xi)\,u_n(\xi)\cos\sqrt{\lambda_n}\,(x-\xi-h)\,d\xi-$$

$$-\int_x^{x+h}q(\xi)\,u_n(\xi)\cos\sqrt{\lambda_n}\,(\xi-x-h)\,d\xi\Big]\Big|\,dh\leqq$$

$$\leqq\sum_{0\leqq\sqrt{\lambda_n}\leqq1}+\sum_{\sqrt{\lambda_n}>1}\leqq C\cdot\mu\cdot\|q\|_{L^1(K_{2R_0})}+C\cdot\mu\cdot\|q\|_{L^1(K_{2R_0})}\cdot\sum_{\sqrt{\lambda_n}>1}\frac{1}{\lambda_n}<\infty.$$

Repeating the argument given above, it is easy to see that for the proof of the existence of $\frac{d^2}{dx^2}h_0(x)$ it is enough to prove the uniform (in x on K) convergence of the series

$$F(x,\mu)=\sum_{n=1}^{\infty}\Big[\int_{y_1}^{y_2}\Big(\int_a^y u_n(\xi)\,d\xi\Big)dy\Big]S_{R_0}\Big(\frac{\partial^2}{\partial x^2}I_n(x,R)\Big).\tag{25}$$

It is easy to see that

(26)

$$\frac{\partial^2}{\partial x^2}I_n(x,R)=-\sqrt{\lambda_n}\int_0^R\frac{\sin\mu h}{h}\Big[\int_{x-h}^{x+h}q(\xi)\,u_n(\xi)\sin\sqrt{\lambda_n}\,(|x-\xi|-h)\,d\xi\Big]\,dh+$$

$$+\int_0^R\frac{\sin\mu h}{h}\big[2q(x)\,u_n(x)\cos\sqrt{\lambda_n}\,h-q(x+h)\,u_n(x+h)-q(x-h)\,u_n(x-h)\big]\,dh.$$

Changing the order of the integrals in the first term, further using the Tichmarsh-formula

$$\frac{u_n(x+h)+u_n(x-h)}{2}=$$

$$=\begin{cases}u_n(x)\cos\sqrt{\lambda_n}\,h-\dfrac{1}{2}\displaystyle\int_{x-h}^{x+h}q(\xi)\,u_n(\xi)\frac{\sin\sqrt{\lambda_n}\,(|x-\xi|-h)}{\sqrt{\lambda_n}}\,d\xi & \text{if}\quad\lambda_n\neq0\\ u_n(x)-\dfrac{1}{2}\displaystyle\int_{x-h}^{x+h}q(\xi)\,u_n(\xi)(|x-\xi|-h)\,d\xi & \text{if}\quad\lambda_n=0\end{cases}$$

in the second term on the right hand side of (26) we get the relation

$$F(x,\mu)=-\sum_{n=1}^{\infty}\Big[\int_{y_1}^{y_2}\Big(\int_a^y u_n(\xi)\,d\xi\Big)dy\Big]\cdot$$

$$\cdot\sqrt{\lambda_n}\cdot S_{R_0}\int_{x-R}^{x+R}q(\xi)\,u_n(\xi)\Big[\int_{|x-\xi|}^{R}\frac{\sin\mu h\sin\sqrt{\lambda_n}\,(|x-\xi|-h)}{\sqrt{\lambda_n}}\,dh\Big]\,d\xi+$$

$$+\sum_{n=1}^{\infty}\Big[\int_{y_1}^{y_2}\Big(\int_a^y u_n(\xi)\,d\xi\Big)dy\Big]\cdot$$

$$\cdot S_{R_0}\Big\{\int_0^R\frac{\sin\mu h}{h}\big[(q(x)-q(x-h))\,u_n(x-h)-(q(x+h)-q(x))\,u_n(x+h)\big]\,dh\Big\}+$$

$$+\sum_{n=1}^{\infty}\Big[\int_{y_1}^{y_2}\Big(\int_a^y u_n(\xi)\,d\xi\Big)dy\Big]\frac{q(x)}{\sqrt{\lambda_n}}S_{R_0}\int_{x-R}^{x+R}q(\xi)\,u_n(\xi)\times$$

$$\times\Big[\int_{|x-\xi|}^{R}\frac{\sin\mu\sin\sqrt{\lambda_n}\,(|x-\xi|-h)}{h}\,dh\Big]\,d\xi=F_1\{x,\mu)+F_2(x,\mu)+F_3(x,\mu).$$

In what follows we need the estimates

$$\Big|\int_{|x-\xi|}^{R}\frac{\sin\mu h\sin\sqrt{\lambda_n}\,(|x-\xi|-h)}{h}\,dh\Big|\leqq\frac{D_1}{|x-\xi|^{\delta}}\quad(0<\delta<1)\tag{27}$$

$$\Big|\int_{|x-\xi|}^{R}\frac{\sin\mu h\sin\sqrt{\lambda_n}\,(|x-\xi|-h)}{h}\,dh\Big|\leqq\frac{D_2}{|x-\xi|^{\delta}\cdot\big|\sqrt{\lambda_n}-\mu\big|^{\delta/2}}\tag{28}$$

$$\big(0<\delta<1,\ |\sqrt{\lambda_n}-\mu|\geqq1\big).$$

These estimates were given and used in [6] without proof. We shall give a proof for them here. It is enough to prove (28), because the estimate (27) is trivial. To this end, first remark that

$$I=\int_{|x-\xi|}^{R}\frac{\sin\mu h\sin\sqrt{\lambda_n}\,(|x-\xi|-h)}{h}\,dh=\tag{29}$$

$$=\sin\sqrt{\lambda_n}|x-\xi|\cdot\int_{|x-\xi|}^{R}\frac{\sin\mu h\cos\sqrt{\lambda_n}\,h}{h}\,dh-\cos\sqrt{\lambda_n}|x-\xi|\cdot\int_{|x-\xi|}^{R}\frac{\sin\mu h\sin\sqrt{\lambda_n}\,h}{h}\,dh=$$

$$=\frac{1}{2}\sin\sqrt{\lambda_n}|x-\xi|\Big[\int_{|x-\xi|}^{R}\frac{\sin(\mu+\sqrt{\lambda_n})\,h}{h}\,dh+\int_{|x-\xi|}^{R}\frac{\sin(\mu-\sqrt{\lambda_n})\,h}{h}\,dh\Big]-$$

$$-\frac{1}{2}\cos\sqrt{\lambda_n}|x-\xi|\Big[\int_{|x-\xi|}^{R}\frac{\cos(\mu-\sqrt{\lambda_n})\,h}{h}\,dh-\int_{|x-\xi|}^{R}\frac{\cos(\mu+\sqrt{\lambda_n})\,h}{h}\,dh\Big].$$

We estimate only the integral

$$\int_{|x-\xi|}^{R} \frac{\sin\left(\mu-\sqrt{\lambda_n}\right)h}{h}\,dh.$$

The estimate of the other terms on the right hand side of (29) goes analogously. Fix $0<\varepsilon<1$ arbitrarily.

First consider the case

$$|x-\xi|\left(\mu-\sqrt{\lambda_n}\right)^{\varepsilon} \leqq R.$$

Then we have

$$\int_{|x-\xi|}^{R} \frac{\sin\left(\mu-\sqrt{\lambda_n}\right)h}{h}\,dh = \int_{|x-\xi|}^{R/(\mu-\sqrt{\lambda_n})^{\varepsilon}} + \int_{R/(\mu-\sqrt{\lambda_n})^{\varepsilon}}^{R}.$$

Using the Theorem of Bonne we get

$$\int_{|x-\xi|}^{R/(\mu-\sqrt{\lambda_n})^{\varepsilon}} \frac{\sin\left(\mu-\sqrt{\lambda_n}\right)h}{h}\,dh = \int_{|x-\xi|(\mu-\sqrt{\lambda_n})}^{R\cdot(\mu-\sqrt{\lambda_n})^{1-\varepsilon}} \frac{\sin y}{y}\,dy =$$

$$= \frac{1}{|x-\xi|^{\delta}\left(\mu-\sqrt{\lambda_n}\right)^{\delta}} \int_{|x-\xi|(\mu-\sqrt{\lambda_n})}^{\eta} \frac{\sin y}{y^{1-\delta}}\,dy,$$

where $y=\left(\mu-\sqrt{\lambda_n}\right)h$, $0<\delta<1$, $|x-\xi|\left(\mu-\sqrt{\lambda_n}\right)\leqq\eta<\dfrac{R}{\left(\mu-\sqrt{\lambda_n}\right)^{\varepsilon}}$. Hence

$$\left|\int_{|x-\xi|}^{R/(\mu-\sqrt{\lambda_n})^{\varepsilon}} \frac{\sin\left(\mu-\sqrt{\lambda_n}\right)h}{h}\,dh\right| \leqq \frac{1}{|x-\xi|^{\delta}\left(\mu-\sqrt{\lambda_n}\right)^{\delta}}\cdot \int_{|x-\xi|(\mu-\sqrt{\lambda_n})}^{\eta} \frac{dy}{y^{1-\delta}} \leqq$$

$$\leqq \frac{1}{|x-\xi|^{\delta}\left(\mu-\sqrt{\lambda_n}\right)^{\delta}} \int_{|x-\xi|}^{\eta} \frac{dy}{y^{1-\delta}} = \frac{1}{\delta\,|x-\xi|^{\delta}\left(\mu-\sqrt{\lambda_n}\right)^{\delta}}\left[\eta^{\delta}-|x-\xi|^{\delta}\right] \leqq$$

$$\leqq \frac{R^{\delta}\left(\mu-\sqrt{\lambda_n}\right)^{(1-\varepsilon)\delta}}{|x-\xi|^{\delta}\left(\mu-\sqrt{\lambda_n}\right)^{\delta}} = \frac{R^{\delta}}{|x-\xi|^{\delta}\left(\mu-\sqrt{\lambda_n}\right)^{\varepsilon\delta}}.$$

(We assumed that $\left|\mu-\sqrt{\lambda_n}\right|\geqq 1$. One can investigate the case $\left|\sqrt{\lambda_n}-\mu\right|\leqq 1$ in a similar way.) On the other hand we have

$$\int_{R(\mu-\sqrt{\lambda_n})^{-\varepsilon}}^{R} \frac{\sin\left(\mu-\sqrt{\lambda_n}\right)h}{h}\,dh = \int_{R(\mu-\sqrt{\lambda_n})^{1-\varepsilon}}^{R(\mu-\sqrt{\lambda_n})} \frac{\sin y}{y}\,dy = \int_{R(\mu-\sqrt{\lambda_n})^{1-\varepsilon}}^{\infty} - \int_{R(\mu-\sqrt{\lambda_n})}^{\infty}$$

and

$$\left|\int_{R(\mu-\sqrt{\lambda_n})^{-\varepsilon}}^{R} \frac{\sin\left(\mu-\sqrt{\lambda_n}\right)h}{h}\,dh\right| \leqq \left|\int_{R(\mu-\sqrt{\lambda_n})^{1-\varepsilon}}^{\infty} \frac{\sin y}{y}\,dy\right| + \left|\int_{R(\mu-\sqrt{\lambda_n})}^{\infty} \frac{\sin y}{y}\,dy\right| =$$

$$= C/R\left(\mu-\sqrt{\lambda_n}\right)^{1-\varepsilon} + C/R\left(\mu-\sqrt{\lambda_n}\right) \leqq C/R\left(\mu-\sqrt{\lambda_n}\right)^{1-\varepsilon} \quad (0<\varepsilon<1).$$

Now consider the case

$$|x-\xi|\cdot\left(\mu-\sqrt{\lambda_n}\right)^{\varepsilon} > R.$$

In this case we have

$$\frac{1}{|x-\xi|} < \frac{\left(\mu-\sqrt{\lambda_n}\right)^{\varepsilon}}{R}$$

and hence

$$\left|\int_{|x-\xi|}^{R} \frac{\sin\left(\mu-\sqrt{\lambda_n}\right)h}{h}\,dh\right| \leqq \frac{C}{|x-\xi|\left(\mu-\sqrt{\lambda_n}\right)} < \frac{C}{R\left(\mu-\sqrt{\lambda_n}\right)^{1-\varepsilon}}.$$

Summarising our estimates and choosing $\varepsilon=1/2$ we obtain

$$\left|\int_{|x-\xi|}^{R} \frac{\sin\left(\mu-\sqrt{\lambda_n}\right)h}{h}\,dh\right| \leqq \frac{C}{|x-\xi|^{\delta}\left(\mu-\sqrt{\lambda_n}\right)^{\delta/2}}$$

and (28) is proved.

REMARK. In (27) and (28) the constants D_1 and D_2 do not depend on μ, $\sqrt{\lambda_n}$, x and ξ.

If r and $\delta>0$ is chosen so that $\dfrac{1}{p}+\dfrac{1}{r}=1$ and $\delta\cdot r<1$, then we have

$$\int_{x-R}^{x+R} \frac{|q(\xi)|}{|x-\xi|^{\delta}}\,d\xi \leqq \|q\|_{L_p(G)}\cdot\left[\frac{(4R_0)^{1-\delta r}}{1-\delta r}\right]^{1/r} = C(R_0). \tag{30}$$

REMARK. The class $C^{(\alpha)}(G)$ $(0<\alpha<1)$ is defined as the set of functions f defined on G such that for every compact subset K of G there exists a constant $\Pi(K)$ by which the estimate

$$|f(x')-f(x'')| \leqq \Pi(K)|x'-x''|^{\alpha} \tag{31}$$

holds for every $x', x''\in K$. Now we estimate the function given in (25). The estimate of F_1 is analogous to that of the series (49) in [3], using the estimates (5), (6), (9), (28), (30). Consider the function F_2 in the form

$$F_2 = \sum_{0\leqq\sqrt{\lambda_n}\leqq 1} + \sum_{\sqrt{\lambda_n}>1}.$$

An easy calculation shows (using (31)) that the estimates

$$\sum_{\sqrt{\lambda_n}>1}\left|\int_{y_1}^{y_2}\left(\int_{a}^{y} u_n(\xi)\,d\xi\right)dy\right|\cdot$$

$$\cdot\left|S_{R_0}\int_{0}^{R}\frac{\sin\mu h}{h}\left[\left(q(x)-q(x-h)\right)u_n(x-h)-\left(q(x+h)-q(x)\right)u_n(x+h)\right]dh\right| \leqq$$

$$\leqq C\cdot\sum_{\sqrt{\lambda_n}>1}\frac{1}{\lambda_n}S_{R_0}\left\{\int_{0}^{R}\frac{1}{h}\cdot 2\Pi(K_{2R_0})h^{\alpha}\,dh\right\} \leqq 2C\Pi(K_{2R_0})\frac{(2R_0)^{\alpha}}{\alpha}\sum_{\sqrt{\lambda_n}>1}\frac{1}{\lambda_n}<\infty$$

hold and hence

$$|F_2(x,\mu)| \leqq C(K) \quad (x\in K,\ \mu\geqq 1,\ a\leqq y_1\leqq y_2\leqq b) \tag{32}$$

follows. Now consider the function $F_3(x,\mu)$ in the form

$$F_3(x,\mu)=\sum_{0\leqq\sqrt{\lambda_n}\leqq 1}{}_n+\sum_{\sqrt{\lambda_n}>1}{}_n.$$

Using the estimates (5), (6), (9), (27), (30),

$$\sum_{\sqrt{\lambda_n}>1}\left|\int_{y_1}^{y_2}\Big(\int_a^y u_n(\xi)\,d\xi\Big)\,dy\left|\frac{|q(x)|}{\sqrt{\lambda_n}}\right|S_{R_0}\int_{x-R}^{x+R}q(\xi)\,u_n(\xi)\times\right.$$

$$\left.\times\Big[\int_{|x-\xi|}^{R}\frac{\sin\mu h\sin\sqrt{\lambda_n}(|x-\xi|-h)}{h}\,dh\Big]\,d\xi\right|\leqq\max_{x\in K}|q(x)|\cdot C(R_0)\cdot\sum_{\sqrt{\lambda_n}>1}\frac{1}{\lambda_n^{3/2}}<\infty$$

follows and

$$|F_3(x,\mu)|\leqq C(K)\quad(x\in K,\ \mu\geqq 1,\ a\leqq y_1\leqq y_2\leqq b)\tag{33}$$

is proved.

From the estimates for F_1, F_2 and F_3 given above it follows: the derivative $h_6''(x)$ exists and the equality

$$\frac{d^2}{dx^2}h_6(x)=\frac{1}{\pi}\sum_{n=1}^{\infty}\Big[\int_{y_1}^{y_2}\Big(\int_a^y u_n(\xi)\,d\xi\Big)\,dy\Big]\cdot S_{R_0}\frac{\partial^2}{\partial x^2}I_n(x,R)\tag{34}$$

holds. Similarly, to prove the existence of the derivative $h_7''(x)$ and the relation

$$\frac{d^2}{dx^2}h_7(x)=\frac{1}{\pi}\sum_{n=1}^{\infty}\Big[\int_{y_1}^{y_2}\Big(\int_a^y u_n(\xi)\,d\xi\Big)\,dy\Big]\cdot S_{R_0}\frac{\partial^2}{\partial x^2}I_n^*(x,R)\quad\Big(\overset{\text{def}}{=\!=}H\Big)\tag{35}$$

$$\Big(\text{where}\quad I_n^*(x,R)=\frac{\sin\mu R}{R}\int_0^R\Big[\int_{x-h}^{x+h}q(\xi)\,u_n(\xi)\frac{\sin\sqrt{\lambda_n}(|x-\xi|-h)}{\sqrt{\lambda_n}}\,d\xi\Big]\,dh\Big),$$

it is enough to prove the uniform convergence in x (on K) of the series H. To this end, consider the sum H in the form

$$\begin{aligned}H(x,\mu)=&\frac{1}{\pi}\sum_{n=1}^{\infty}\Big[\int_{y_1}^{y_2}\Big(\int_a^y u_n(\xi)\,d\xi\Big)\,dy\Big]\cdot S_{R_0}-\\&-\sqrt{\lambda_n}\frac{\sin\mu R}{R}\int_{x-R}^{x+R}q(\xi)\,u_n(\xi)\Big[\int_{|x-\xi|}^{R}\sin\sqrt{\lambda_n}(|x-\xi|-h)\,dh\Big]\,d\xi+\\&+\frac{2}{\pi}\sum_{n=1}^{\infty}\Big[\int_{y_1}^{y_2}\Big(\int_a^y u_n(\xi)\,d\xi\Big)\,dy\Big]\,u_n(x)\,q(x)\cdot S_{R_0}\Big[\frac{\sin\mu R\sin\sqrt{\lambda_n}R}{R\sqrt{\lambda_n}}\Big]-\\&-\frac{1}{\pi}\sum_{n=1}^{\infty}\Big[\int_{y_1}^{y_2}\Big(\int_a^y u_n(\xi)\,d\xi\Big)\,dy\Big]\cdot\\&\cdot S_{R_0}\frac{\sin\mu R}{R}\int_0^R[q(x+h)\,u_n(x+h)-q(x-h)\,u_n(x-h)]\,dh\equiv\\&\equiv\frac{1}{\pi}H_1(x,\mu)+\frac{2}{\pi}H_2(x,\mu)-\frac{1}{\pi}H_3(x,\mu).\end{aligned}\tag{35'}$$

Using the estimates (5), (6) and (9) we obtain

$$\begin{aligned}|H_1(x,\mu)|\leqq&\sum_{0\leqq\sqrt{\lambda_n}\leqq 1}\Big|\int_{y_1}^{y_2}\Big(\int_a^y u_n(\xi)\,d\xi\Big)\,dy\Big|\cdot|\sqrt{\lambda_n}|\cdot\\&\cdot\Big|S_{R_0}\frac{\sin\mu R}{R}\int_{x-R}^{x+R}q(\xi)\,u_n(\xi)\int_{|x-\xi|}^{R}\sin\sqrt{\lambda_n}(|x-\xi|-h)\,dh\,d\xi\Big|+\\&+\sum_{\sqrt{\lambda_n}>1}\Big|\int_{y_1}^{y_2}\Big(\int_a^y u_n(\xi)\,d\xi\Big)\,dy\Big|\cdot\Big|S_{R_0}\frac{\sin\mu R}{R}\int_{x-R}^{x+R}q(\xi)\,u_n(\xi)\frac{\cos\sqrt{\lambda_n}(|x-\xi|-R)-1}{\sqrt{\lambda_n}}\,d\xi\Big|\leqq\\&\leqq\tilde{C}(K)+\tilde{C}(K)\cdot\sum_{\sqrt{\lambda_n}>1}\frac{1}{\lambda_n^{3/2}}\leqq CK)\end{aligned}\tag{36}$$

and

$$\begin{aligned}|H_2(x,\mu)|\leqq&\sum_{0\leqq\sqrt{\lambda_n}\leqq 1}\Big|\int_{y_1}^{y_2}\Big(\int_a^y u_n(\xi)\,d\xi\Big)\,dy\Big|\cdot|u_n(x)||q(x)|\cdot\Big|S_{R_0}\frac{\sin\mu R\sin\sqrt{\lambda_n}R}{R\sqrt{\lambda_n}}\Big|+\\&+\sum_{\sqrt{\lambda_n}>1}\Big|\int_{y_1}^{y_2}\Big(\int_a^y u_n(\xi)\,d\xi\Big)\,dy\Big|\cdot|u_n(x)|\cdot|q(x)|\cdot\Big|S_{R_0}\frac{\sin\mu R\sin\sqrt{\lambda_n}R}{R\sqrt{\lambda_n}}\Big|\leqq\\&\leqq\tilde{C}(K)\cdot\max_{x\in K}|q(x)|+\tilde{C}(K)\max_{x\in K}|q(x)|\sum_{\sqrt{\lambda_n}>1}\frac{1}{\lambda_n^{3/2}}\leqq C(K).\end{aligned}\tag{37}$$

Using (5), (9) and (10) we obtain

$$\begin{aligned}|H_3(x,\mu)|\leqq&\sum_{0\leqq\sqrt{\lambda_n}\leqq 1}\Big|\int_{y_1}^{y_2}\Big(\int_a^y u_n(\xi)\,d\xi\Big)\,dy\Big|\cdot S_{R_0}\Big|\frac{\sin\mu R}{R}\Big|\cdot\\&\cdot\int_0^R|q(x+h)\,u_n(x+h)+q(x-h)\,u_n(x-h)|\,dh+\sum_{\sqrt{\lambda_n}>1}\Big|\int_{y_1}^{y_2}\Big(\int_a^y u_n(\xi)\,d\xi\Big)\,dy\Big|\cdot\\&\cdot S_{R_0}\Big|\frac{\sin\mu R}{R}\Big|\cdot\int_0^R|q(x+h)\,u_n(x+h)+q(x-h)\,u_n(x-h)|\,dh\leqq\\&\leqq\tilde{C}(K)+\tilde{C}(K)\cdot\sum_{\sqrt{\lambda_n}>1}\frac{1}{\lambda_n}\leqq C(K).\end{aligned}\tag{38}$$

From (36), (37) and (38) the estimate

$$|H(x,\mu)|\leqq C(K)\quad(x\in K,\ \mu\geqq 1,\ a\leqq y_1\leqq y_2\leqq b)\tag{39}$$

follows and (35) is also proved. Thus we have proved that the derivatives $g''(x)$ and $h_i''(x)$ $(i=1,2,\ldots,7)$ exist and

$$g''(x)=h''(x)\Big(\equiv\sum_{i=1}^{7}h_i''(x)\Big).$$

Taking into consideration (18), (23), (30), (35), we can write the last relation in the following form:

$$(40)\quad \int_{y_1}^{y_2}\left(\int_a^y \frac{\partial^2}{\partial x^2}[S_{R_0}v_R(x,\xi,\mu)-\theta(x,\xi,\mu)]\,d\xi\right)dy = -F(x,y_1,y_2,\mu,R_0)+$$

$$+\frac{\partial^2}{\partial x^2}\left(\int_{y_1}^{y_2}\left(\int_a^y S_{R_0}\omega_R(x,\xi,\mu)\,d\xi\right)dy\right)+\frac{1}{2}\sum_{\sqrt{\lambda_n}=\mu}\left[\int_{y_1}^{y_2}\left(\int_a^y u_n(\xi)\,d\xi\right)dy\right]\cdot\frac{d^2}{dx^2}u_n(x)-$$

$$-\sum_{0\leqq\sqrt{\lambda_n}\leqq 1}\left[\int_{y_1}^{y_2}\left(\int_a^y u_n(\xi)\,d\xi\right)dy\right]\cdot S_{R_0}[I^{\mu}_{\sqrt{\lambda_n}}(R)]\cdot\frac{d^2}{dx^2}u_n(x)-$$

$$-\frac{2}{\pi}\sum_{0\leqq\sqrt{\lambda_n}\leqq 1}\left[\int_{y_1}^{y_2}\left(\int_a^y u_n(\xi)\,d\xi\right)dy\right]\cdot S_{R_0}\left[\frac{\sin\mu R\sin\sqrt{\lambda_n}R}{\sqrt{\lambda_n}R}\right]\frac{d^2}{dx^2}u_n(x)-$$

$$-\sum_{\sqrt{\lambda_n}>1}\left[\int_{y_1}^{y_2}\left(\int_a^y u_n(\xi)\,d\xi\right)dy\right]\cdot\frac{\mu}{\sqrt{\lambda_n}}S_{R_0}[K^{\mu}_{\sqrt{\lambda_n}}(R)]\cdot\frac{d^2}{dx^2}u_n(x)-$$

$$-\frac{1}{\pi}\{F_1(x,\mu)-F_2(x,\mu)-F_3(x,\mu)\}+H(x,\mu).$$

Now we prove (14). Denote S_i ($i=1, 2, \ldots, 10$) the i-th term on the right hand side of (40) and estimate these terms below.

$$(41)\qquad |S_1(x)|\leqq C(K)\quad (x\in K,\ \mu\geqq 1,\ a\leqq y_1\leqq y_2\leqq b)$$

was given in (19);

$$(42)\qquad |S_2(x)|\leqq C(K)\quad (x\in K,\ \mu\geqq 1,\ a\leqq y_1\leqq y_2\leqq b)$$

follows from the proof of (20) and (21). One can prove analogous estimates for S_3, S_4, S_5 by the method applied in [6] for the proof of (45), using also the estimates (8) and (9) of the present work. For S_6 and S_7 one can prove the analogous estimates by the method applied in [3] in the proof of the similar estimates for the functions S_5 and S_6 given in [3]. For S_8, S_9 and S_{10} the corresponding estimates were proved above. Lemma 3 (i.e. the estimate (14)) is proved.

One can prove similarly to Lemma 3 the following

LEMMA 3. *With the assumptions of Lemma* 3 *the estimate*

$$(43)\qquad \left|\int_{y_1}^{y_2}\left(\int_y^b \frac{\partial^2}{\partial x^2}[S_{R_0}v_R(x,\xi,\mu)-\theta(x,\xi,\mu)]\,d\xi\right)dy\right|\leqq C(K)$$

$$(x\in K,\ \mu\geqq 1,\ a\leqq y_1\leqq y_2\leqq b)$$

holds.

§3. Proof of Theorems 1 and 2

First we prove Theorem 1. Using (17), it follows that the function $\left[\frac{\partial^2}{\partial x^2}S_{R_0}v_R(x,y,\mu)\right]\cdot f(y)$ is Lebesque integrable on the interval (a, b). Introduce the notation

$$(44)\qquad S_\mu(f,x)=\int_a^b \frac{\partial^2}{\partial x^2}S_{R_0}v_R(x,y,\mu)\cdot f(y)\,dy,$$

and consider the functions

$$(45)\qquad g(y)=\frac{f(b)}{b-a}(y-a),\quad h(y)=f(y)-g(y)$$

and

$$(46)\quad \frac{d^2}{dx^2}\sigma_\mu(f,x)-S_\mu(f,x)=\int_a^b\frac{\partial^2}{\partial x^2}[\theta(x,y,\mu)-S_{R_0}v_R(x,y,\mu)]\cdot g(y)\,dy+$$

$$+\int_a^b\frac{\partial^2}{\partial x^2}[\theta(x,y,\mu)-S_{R_0}v_R(x,y,\mu)]\cdot h(y)\,dy.$$

The functions in the last integrals are continuous in x on the set $[a, b]\setminus\{R_0, 2R_0\}$, hence instead of (46) we can write

(47)

$$\frac{d^2}{dx^2}\sigma_\mu(f,x)-S_\mu(f,x)=\int_a^b g(y)\frac{\partial}{\partial y}\left[\int_y^b\frac{\partial^2}{\partial x^2}(S_{R_0}v_R(x,\xi,\mu)-\theta(x,\xi,\mu))\,d\xi\right]dy+$$

$$+\int_a^b h(y)\cdot\frac{\partial}{\partial y}\left[\int_a^y\frac{\partial^2}{\partial x^2}(\theta(x,\xi,\mu)-S_{R_0}v_R(x,\xi,\mu))\,d\xi\right]dy.$$

Now define the functions $i(y)$ and $j(y)$ on the interval $[a, b]$ in the following way:

$$(48)\qquad i(y)=\frac{h'(b)}{b-a}(y-a),\quad j(y)=h'(y)-i(y)$$

and write (47) in the following form:

$$\frac{d^2}{dx^2}\sigma_\mu(f,x)-S_\mu(f,x)=\int_a^b\frac{\partial}{\partial y}\left[\int_a^y\int_t^b\frac{\partial^2}{\partial x^2}(S_{R_0}v_R(x,\xi,\mu)-\theta(x,\xi,\mu))\,d\xi\,dt\right]g'(y)\,dy-$$

$$(49)\qquad -\int_a^b i(y)\frac{\partial}{\partial y}\left[\int_y^b\left(\int_a^t\frac{\partial^2}{\partial x^2}(\theta(x,\xi,\mu)-S_{R_0}v_R(x,\xi,\mu))\,d\xi\right)dt\right]dy+$$

$$+\int_a^b j(y)\frac{\partial}{\partial y}\left[\int_a^y\left(\int_a^t\frac{\partial^2}{\partial x^2}(\theta(x,\xi,\mu)-S_{R_0}v_R(x,\xi,\mu))\,d\xi\right)dt\right]dy.$$

Integrating by parts and using (45) and (48) we obtain

$$\frac{d^2}{dx^2}\sigma_\mu(f,x)-S_\mu(f,x)=g'(b)\int_a^b\left(\int_t^b\frac{\partial^2}{\partial x^2}\left(S_{R_0}v_R(x,\xi,\mu)-\theta(x,\xi,\mu)\right)d\xi\right)dt+$$

$$(50)\qquad +\int_a^b\left[\int_y^b\left(\int_a^t\frac{\partial^2}{\partial x^2}\left(\theta(x,\xi,\mu)-S_{R_0}v_R(x,\xi,\mu)\right)d\xi\right)dt\right]i'(y)\,dy-$$

$$-\int_a^b\left[\int_a^y\left(\int_a^t\frac{\partial^2}{\partial x^2}\left(\theta(x,\xi,\mu)-S_{R_0}v_R(x,\xi,\mu)\right)d\xi\right)dt\right]j'(y)\,dy.$$

From the last equality we obtain (using (14) and (43))

$$(51)\qquad \left|\frac{d^2}{dx^2}\sigma_\mu(f,x)-S_\mu(f,x)\right|\leqq\left|\frac{f(b)}{b-a}\right|\cdot C_1'(K)+(|f'(b)|+|g'(b)|)\,C_1(K)+$$

$$+(|f'(b)|+|g'(b)|)\,C_1(K)+C_1(K)\cdot\|f''\|_{L^1(G)}=$$

$$=\frac{|f(b)|}{b-a}\left(C_1'(K)+2C_1(K)\right)+2C_1(K)\cdot|f'(b)|+C_1(K)\|f''\|_{L^1(G)},$$

uniformly in x on the compact set K and uniformly in μ on $(0,\infty)$. It is easy to see that the quantities $|f(b)|$, $|f'(b)|$ and $\|f''\|_{L^1(G)}$ can be estimated by the norm $\|f\|_{W_1^2(G)}$ and from (51) we obtain

$$(52)\qquad \left|\frac{d^2}{dx^2}\sigma_\mu(f,x)-S_\mu(f,x)\right|\leqq C(K)\quad(x\in K,\ \mu\geqq 0),$$

similarly

$$(53)\qquad \left|\frac{d^2}{dx^2}\hat{\sigma}_\mu(f,x)-S_\mu(f,x)\right|\leqq C(K)\quad(x\in K,\ \mu\geqq 0),$$

and consequently

$$(54)\qquad \left|\frac{d^2}{dx^2}\sigma_\mu(f,x)-\frac{d^2}{dx^2}\hat{\sigma}_\mu(f,x)\right|\leqq C(K)\quad(x\in K,\ \mu\geqq 0).$$

The estimate (3) is proved. Now we prove: (3) cannot be improved in the sense that $o(1)$ $(\mu\to\infty)$ cannot be written on the right hand side of (3).

Suppose this is not the case, i.e. we have $o(1)$ $(\mu\to\infty)$ on the right hand side of (3).

Suppose $G=(0,\pi)$, $q\equiv 0$, $\hat{q}=1/2$. Then $u_n(x)=\sqrt{2}\cos(nx)$, $\lambda_n=n$, further $\hat{u}_n(x)=\sqrt{2}\cos(nx)$, $\hat{\lambda}_n=n+1/2$ $(n=0,1,2,\ldots)$. From the indirect assumption we get

$$\sigma''_{n+1/4}(f,x)-\hat{\sigma}''_{n+1/4}(f,x)=o(1),\quad \sigma''_{n-1/4}(f,x)-\hat{\sigma}''_{n-1/4}(f,x)=o(1)\quad(n\to\infty)$$

and hence

$$\sigma''_{n+1/4}(f,x)-\sigma''_{n-1/4}(f,x)=-\frac{n^2}{\pi}\cos nx\int_0^\pi f(t)\cos nt\,dt=o(1)$$
$$(n\to\infty,\quad x\in K\subset G),$$

for every $f\in W_1^2(0,\pi)$. If K is a closed subinterval of G with mes $K>0$, then we have the lower estimate $\left|\int_K\cos nx\,dx\right|>c>0$ $(n=0,1,\ldots)$ by which

$$(55)\qquad \int_0^\pi f(t)\cos nt\,dt=o(1/n^2)\quad(n\to\infty),$$

for every $f\in W_1^2(G)$. Now let $0\leqq g\in L^1(G)$ be arbitrary and $f(x)\overset{\text{def}}{=}\int_0^x\int_0^\xi g(t)\,dt\,d\xi$. Obviously $f\in W_1^2(G)$, $f'(0)=0$, $f'(\pi)\neq 0$, further using (55)

$$(56)\qquad \int_0^\pi f'(t)\sin nt\,dt=o(1/n)\quad(n\to\infty).$$

On the other hand, integrating by parts we obtain

$$\left|\int_0^\pi f'(t)\sin nt\,dt\right|\geqq\left|f'(\pi)\frac{\cos n\pi}{n}\right|-\left|\int_0^\pi f''(t)\frac{\cos nt}{n}dt\right|=$$

$$=\left|f'(\pi)\frac{\cos n\pi}{n}\right|-\left|\int_0^\pi g(t)\frac{\cos nt}{n}dt\right|\geqq C_f\cdot\frac{1}{n}$$

and this contradicts (56). Theorem 1 is proved.

Remark 1. By the method of the proof of the exactness of the estimate (3) it ollows that in the case of a general boundary value problem (or in a more general case) we can write only $O(\mu)$ on the right hand side of (3). We omit the details.

Now we prove Theorem 2. To this end, let $\varepsilon>0$ be an arbitrarily small number. It is well known (cf. [5]) that for every $f\in W_1^2(G)$ with compact support there exists a function $g\in C_0^\infty(G)$ such that

$$\|f-g\|_{W_1^2(G)}<\varepsilon.$$

We prove first that $\sigma''_\mu(g,x)$ tends to $g''(x)$ as $\mu\to\infty$ uniformly in x on every compact subset of G. To this end, it is enough to prove the uniform convergence of the sequence σ''_μ, because in [3] we proved that $\sigma'_\mu(g,x)\to g'(x)$ uniformly in x on every compact subset of G. In the following we shall suppose that $q''(x)\in L^1(G)$. In this case it is easy to prove that

$$(57)\qquad g_n\overset{\text{def}}{=}\int_G g(x)\,u_n(x)\,dx=\frac{1}{\lambda_n^2}\int_G h(x)\,u_n(x)\,dx\overset{\text{def}}{=}\frac{1}{\lambda_n^2}h_n\quad(\lambda_n\neq 0)$$

where

$$h(x)=L^2g(x)=[-g''(x)+q(x)g(x)]''+q(x)[-g''(x)+q(x)g(x)]\in L^1(G).$$

Using the estimates (5), (6), (7) and the relation (57) we obtain

$$\sum_{n=1}^\infty|g_n|\left|\frac{d^2}{dx^2}u_n(x)\right|=\sum_{0\leqq\sqrt{\lambda_n}\leqq 1}\left|\int_G g(x)\,u_n(x)\,dx\right|\cdot\left|\frac{d^2}{dx^2}u_n(x)\right|+$$

$$+\sum_{\sqrt{\lambda_n}>1}\frac{1}{\lambda_n^2}h_n\left|\frac{d^2}{dx^2}u_n(x)\right|\leqq\tilde{C}(K)+\tilde{C}(K)\sum_{\sqrt{\lambda_n}>1}\frac{1}{\lambda_n}\leqq C(K),$$

and the uniform convergence of the sequence $\sigma''_\mu(g, x)$ is proved. So there exists a number $\mu_0>0$ such that

$$|\sigma''_\mu(g, x)-g''(x)| < \varepsilon/4 \quad (x\in K,\ \mu \geqq \mu_0). \tag{58}$$

Similarly

$$\left|\frac{d^2}{dx^2}\hat{\sigma}_\mu(g, x)-\frac{d^2}{dx^2}g(x)\right| \leqq \frac{\varepsilon}{4} \quad (x\in K,\ \mu \geqq \mu_0^*).$$

Summarising our estimates we obtain

$$|\sigma''_\mu(f, x)-\hat{\sigma}''_\mu(f, x)| \leqq |\sigma''_\mu(f-g, x)-\hat{\sigma}''_\mu(f-g, x)|+|\sigma''_\mu(g, x)-\hat{\sigma}''_\mu(g, x)| \leqq$$

$$\leqq \tilde{C}(K)\cdot\|f-g\|_{W_1^2(G)}+|\sigma''_\mu(g, x)-g''(x)|+|\hat{\sigma}''_\mu(g, x)-g''(x)| \leqq C(K)\cdot\varepsilon,$$

and (4) is proved. Now we show that (4) cannot be improved in the sense that we can not assert $o(\alpha(\mu))$ on the right hand side of (4) where $\alpha(\mu)$ is such a function for which $\lim \alpha(\mu)=0$ as $\mu\to\infty$. Then the argument given in the proof of Theorem 1 gives the estimate

$$\int_0^\pi f(t)\cos nt\,dt = O\left(\frac{\alpha(n)}{n^2}\right) \tag{59}$$

for every $f\in W_1^2(G)$ having compact support in G.

Let $g\in L^1(G)$ be arbitrary and define

$$f(x) = \int_0^x \left(\int_0^\xi g(t)\,dt\right)d\xi+Cx,$$

where

$$C = -\int_0^\pi g(t)\,dt.$$

Obviously $f(0)=f'(\pi)=0$ further $f\in W_1^2(G)$. Integrating twice, we obtain from (59) for the just defined function:

$$\int_0^\pi g(t)\cos nt\,dt = O(\alpha(n)) \quad (n = 1, 2, \ldots) \tag{60}$$

for every $g\in L^1(G)$. Now we prove that this is not the case. To this end consider an infinite sequence $\{n_k\}$ of natural numbers, such that

$$\alpha(n_k) \neq 0, \quad \text{for} \quad k = 1, 2, \ldots.$$

Suppose that (60) is fulfilled. Consider the operator $A: L^1(G)\to c_\infty$ (c_∞ is the Banach space of the bounded sequences with supremun norm) defined as

$$A(g) = \left\{\frac{c_g(n_k)}{\alpha(n_k)}\right\}_{k=1}^{\infty}, \quad c_g(n_k) = \int_0^\pi g(t)\cos n_k t\,dt.$$

According to Banach's closed graph theorem, A is continuous (namely it is easy to prove that A is closed). On the other hand, the sequence

$$g_k(x) = \alpha(n_k)\cdot\cos n_k x$$

tends to 0 in the space $L^1(G)$ but $A(g_k)$ does not tend to 0 in the space c_∞. This is a contradiction and Theorem 2 is proved.

Remark 2. It is possible to generalize Theorem 2 for the case of arbitrary complete orthonormal system of eigenfunctions (not necessarily connected with the second boundary condition).

§4. The case of arbitrary non-negative selfadjoint extension

Let $\{v_n\}$ be an arbitrary fundamental system which corresponds to the arbitrary non-negative selfadjoint extension of the operator $Lv=-v''+q(x)v(x)$. Then instead of (9) we have only the estimate

$$\sum_{|\sqrt{\lambda_n}-\mu|\leqq 1}\left|\int_{y_1}^{y_2}\left(\int_a^y v_n(\xi)\,d\xi\right)dy\right| \leqq \frac{C_3}{\mu^2} \quad (\mu \geqq 1,\ a \leqq y_1 \leqq y_2 \leqq b). \tag{61}$$

Denote $\sigma_\mu(f, x)$ and $\hat{\sigma}_\mu(f, x)$ the spectral expansions of a function f with absolutely continuous derivative on the closed interval $[a, b]=\bar{G}$, corresponding to two arbitrary non-negative selfadjoint extensions of the operators

$$Lu = -u''+q(x)u, \quad \hat{L}u = -u''+\hat{q}(x)u.$$

We assume that

$$q(x)\in C(G)\cap L^p(G), \quad \hat{q}(x)\in C(G)\cap L^p(G) \quad (1 < p < \infty). \tag{62}$$

The method of the proof of Theorem 1 yields (using (61))

Theorem 3. *If for the potentials q and $\hat{q}$ condition (62) is fulfilled then*

$$\frac{d^2}{dx^2}\sigma_\mu(f, x)-\frac{d^2}{dx^2}\hat{\sigma}_\mu(f, x) = O(\mu) \quad (\mu \geqq 1)$$

uniformly in x on every compact subset of G. The order of this estimate cannot be improved.

By the method of the proof of Theorem 2 we get

Theorem 4. *If the conditions of the Theorem 3 are fulfilled and f has a compact support in G further $q''(x)$ and $\hat{q}''(x)$ belong to $L^1(G)$, then*

$$\frac{d^2}{dx^2}\sigma_\mu(f, x)-\frac{d^2}{dx^2}\hat{\sigma}_\mu(f, x) = o(\mu) \quad (\mu \to \infty)$$

uniformly in x on every compact subset of G.

The order of this estimate can be improved (cf. Remark 2), but this needs essential modifications in the proof.

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ÜBER DIE KNESER-KONSTANTE DER DIFFERENTIALGLEICHUNG $(-\Delta)^m u+q(x)u=0$

Von

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Im folgenden soll untersucht werden, unter welchen Bedingungen für die Funktion $q(x)$ die elliptische Differentialgleichung

$$(1)\qquad (-\Delta)^m u+q(x)u=0,\ x=(x_1,\dots,x_n)\in\mathbf{R}^n,\ \Delta=\frac{\partial^2}{\partial x_1^2}+\dots+\frac{\partial^2}{\partial x_n^2},\ m\geqq 1,$$

$$q(x)\in C(\mathbf{R}^n)\quad\text{und reell,}$$

im Raum $\mathbf{R}^n$ oszillatorisch ist oder nicht.

Ein beschränktes Gebiet G des $\mathbf{R}^n$ heißt *Knotengebiet* der Gleichung (1), falls eine nichttriviale Lösung $u=u(x)$ von (1) existiert, die zu dem Raum $W_{2,\mathrm{loc}}^{2m}(G)\cap\cap\mathring{W}_2^m(G)$[1] gehört. Die Gleichung (1) heißt *oszillatorisch,* wenn sie außerhalb jeder im Endlichen gelegenen Kugel ein Knotengebiet besitzt. Sie heißt entsprechend *nichtoszillatorisch* im $\mathbf{R}^n$, wenn eine endliche Kugel im $\mathbf{R}^n$ existiert, so daß außerhalb dieser Kugel kein Knotengebiet von (1) liegt. Was die Differentialgleichung

$$(2)\qquad -u''+q(x)u=0,\quad 0<x<\infty,\quad q(x)\in C(0,\infty),$$

betrifft, so stellt das klassische Resultat von A. Kneser fest, daß die Gleichung (2) nichtoszillatorisch bzw. oszillatorisch ist, je nachdem ob

$$(3)\qquad \varliminf_{x\to\infty} x^2q(x)>-\frac{1}{4}\quad\text{oder}\quad \varlimsup_{x\to\infty} x^2q(x)<-\frac{1}{4}$$

gilt. Steht in (3) des Gleichheitszeichen, so sind beide Fälle möglich. Das Oszillationskriterium von Kneser ist vielfach verallgemeinert worden. Zusammenfassende Darstellungen solcher Oszillationskriterien kann man in den Büchern von I. M. GLAZMAN [3], K. KREITH [5] und C. A. SWANSON [8] finden. Wir weisen an dieser Stelle noch auf die Verallgemeinerung der Bedingungen (3) durch E. HILLE [4] hin. Die

[1] $W_{2,\mathrm{loc}}^{2m}(G)$ und $\mathring{W}_2^m(G)$ sind Sobolevsche Räume. $W_{2,\mathrm{loc}}^{2m}(G)$ enthält alle diejenigen in G definierten (komplexwertigen) Funktionen, deren verallgemeinerte Ableitungen bis zur Ordnung $2m$ auf jeder kompakten Teilmenge von G quadratisch integrierbar sind. $\mathring{W}_2^m(G)$ entsteht aus $C_0^\infty(G)$ durch Vervollständigung in der Norm

$$\|u\|_{W_2^m(G)}=\left(\sum_{0\leqq l_1+\dots+l_n\leqq m}\int_G\left|\frac{\partial^{l_1+\dots+l_n}u}{\partial x_1^{l_1}\dots\partial x_n^{l_n}}\right|^2dx\right)^{1/2}.$$

Differentialgleichung (2) ist nichtoszillatorisch bzw. oszillatorisch, je nachdem ob

$$(4)\qquad \liminf_{x\to\infty} x^2\log^2 x\left[q(x)+\frac{1}{4x^2}\right] > -\frac{1}{4}$$

oder

$$(4)\qquad \limsup_{x\to\infty} x^2\log^2 x\left[q(x)+\frac{1}{4x^2}\right] < -\frac{1}{4}$$

gilt. In der vorliegenden Arbeit werden Oszillationskriterien von der Art (4) für die Gleichung (1) angegeben. Auf Bedingungen vom Kneser-Typ für elliptische Differentialgleichungen höherer Ordnung hat bezüglich der Nichtoszillation der Gleichung auch W. ALLEGRETTO [1], [2] hingewiesen.

SATZ 1. *Es sei $2m \leqq n$ oder $n<2m$, n ungerade.*[2] *Die Gleichung*

$$(-\Delta)^m u+q(x)u=0,\quad x=(x_1,\dots,x_n)\in\mathbf{R}^n,$$

ist nichtoszillatorisch, wenn ein $r_0>0$ existiert, so daß

$$(5)\qquad q(x) > -\left(\sum_{j=0}^{m}\frac{[(2j-1)!!]^2\alpha_j}{4^j\log^{2j}|x|}\right)\frac{1}{|x|^{2m}},\quad |x|>r_0,$$

gilt; dabei sind die Zahlen α_j die Koeffizienten des Polynoms

$$\sum_{j=0}^{m}\alpha_j\xi^j=\left[\xi+\left(\frac{n-2m}{2}\right)^2\right]\left[\xi+\left(\frac{n-2m+4}{2}\right)^2\right]\dots\left[\xi+\left(\frac{n+2m-4}{2}\right)^2\right],\quad \alpha_m=1.$$

Die Gleichung ist oszillatorisch, wenn eine Funktion $\delta(r)$ mit der Eigenschaft

$$\int_2^\infty\frac{\delta(r)}{r\log r}\,dr=-\infty$$

und eine Zahl r_0 existieren, so daß

$$(6)\qquad q(x)\leqq-\left(\sum_{j=0}^{1}\frac{\alpha_j}{4^j\log^{2j}|x|}\right)\frac{1}{|x|^{2m}}+\frac{\delta(|x|)}{|x|^{2m}\log^2|x|},\quad |x|>r_0$$

gilt.

BEWEIS. Wir entnehmen einen Teil der folgenden Betrachtungen einer Arbeit von MIŠIN [7], wo unter einem etwas anderen Aspekt Abschätzungen durchgeführt werden, die für den Beweis des Satzes nutzbar gemacht werden können.[3]

Bei Verwendung des Beltrami-Operators B hat der Laplace-Operator die Form

$$\Delta u(x)=\left(\frac{d^2}{dr^2}+\frac{n-1}{r}\frac{d}{dr}+\frac{B}{r^2}\right)u(x).$$

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[2] Der Fall $n<2m$, n gerade wird später behandelt.

[3] Der Verfasser sieht sich veranlaßt, Teile der Arbeit [7] zu wiederholen, da nach seiner Meinung in [7] Fehler vorhanden sind. So ist z. B. das Lemma, das in [7] ohne Beweis erscheint und das im Theorem 4 benutzt wird, im allgemeinen nicht richtig.

Bezeichnet $\{S_k(\vartheta)\}_{k=1,2,\dots}$ das System der orthonormierten Kugelflächenfunktionen, so kann eine Funktion $u(x)\in C_0^\infty(\mathbf{R}^n\setminus\{0\})$ als Reihe

$$(7)\qquad u(x)=u(r,\vartheta)=\sum_{k=1}^{\infty}S_k(\vartheta)R_k(r)$$

mit

$$R_k(r)=\int_{\omega_n}u(r,\vartheta)\,S_k(\vartheta)\,d\vartheta,\quad \omega_n=\{x\,|\,x\in\mathbf{R}^n \text{ und } |x|=1\},$$

dargestellt werden. Aus (7) folgt

$$(8)\qquad \Delta^m u(x)=\sum_{k=1}^{\infty}S_k(\vartheta)\cdot B_k^m R_k(r),$$

wobei

$$B_k=\frac{d^2}{dr^2}+\frac{n-1}{r}\frac{d}{dr}+\frac{s_k}{r^2}$$

ist. s_k ist der zur Eigenfunktion $S_k(\vartheta)$ von B gehörige Eigenwert. s_k ist für $k>1$ ein mehrfacher Eigenwert; dabei gilt

$$(9)\qquad s_k=-k'(k'+n-2),\quad k'=0,1,2,\dots.$$

$s_1=0$ $(k'=0)$ ist der einzige einfache Eigenwert. Mit (7) und (8) folgt

$$(10)\qquad \int_{\mathbf{R}^n}\Delta^m u\cdot\bar{u}\,dx=\sum_{k=1}^{\infty}\int_0^\infty B_k^m R_k(r)\cdot\overline{R_k(r)}\,r^{n-1}\,dr.$$

Bei der Transformation

$$(11)\qquad t=\log r,\quad R_k(r)=R_k(e^t)=P_k(t)\,e^{(2m-n)t/2},$$

bekommt der Ausdruck $B_k^m R_k(r)$ die Gestalt

$$\left(\sum_{l=0}^{2m}c_{l,k}P_k^{(l)}(t)\right)e^{-(2m+n)t/2},\quad c_{2m,k}=1.$$

Es wird im folgenden gezeigt, daß die Koeffizienten $c_{l,k}$ die Eigenschaften

$$c_{2j-1,k}=0,\ c_{2j,k}=(-1)^{m-j}|c_{2j,k}|,\quad j=0,\dots,m,$$

besitzen. Die Gleichung $B_k^m\hat{R}_k(r)=0$ besitzt als Lösungen Potenzfunktionen

$$r^{\lambda_{1,k}+2\nu},\ r^{\lambda_{2,k}+2\nu},\ \nu=0,1,\dots,m-1,$$

wobei $\lambda_{1,k}$ und $\lambda_{2,k}$ die Nullstellen des Polynoms

$$\lambda^2+(n-2)\lambda+s_k=0$$

sind. Wegen (9) erhält man $\lambda_{1,k}=k'$, $\lambda_{2,k}=2-n-k'$. Die den Potenzfunktionen entsprechenden Lösungen der Gleichung

$$(12)\qquad \sum_{l=0}^{2m}c_{l,k}\hat{P}_k^{(l)}(t)=0$$

heißen also

$$(13)\qquad e^{\pm(k'+(n-2m)/2)t},\ e^{\pm(k'+(n-2m+4)/2)t},\ \dots,\ e^{\pm(k'+(n+2m-4)/2)t}.$$

Die Nullstellen des charakteristischen Polynoms des Differentialausdrucks (12) sind die Faktoren in den Exponenten der Funktionen (13). So folgen für die Koeffizienten $c_{l,k}$ die Eigenschaften

$$c_{2j-1,k}=0,\ \ c_{2j,k}=(-1)^{m-j}|c_{2j,k}|,\quad j=1,\dots,m,$$

und

$$c_{0,k}=(-1)^m\prod_{\nu=0}^{m-1}\left(k'+\frac{n-2m+4\nu}{2}\right)^2.$$

Das charakteristische Polynom des Differentialausdrucks (12) heißt also

$$(14)\quad \sum_{l=0}^{2m}c_{l,k}\lambda^l=\sum_{j=0}^{m}c_{2j,k}\lambda^{2j}=\left[\lambda^2-\left(k'+\frac{n-2m}{2}\right)^2\right]\cdots\left[\lambda^2-\left(k'+\frac{n+2m-4}{2}\right)^2\right].$$

Wird in (10) die Transformation (11) ausgeführt, so folgt

$$(15)\qquad \int_{\mathbf{R}^n}(-\Delta)^m u\bar{u}\,dx=\sum_{k=1}^{\infty}\int_{-\infty}^{\infty}\left(\sum_{j=0}^{m}(-1)^j|c_{2j,k}|P_k^{(2j)}(t)\,\overline{P_k(t)}\right)dt=$$

$$=\sum_{k=1}^{\infty}\int_{-\infty}^{\infty}\left(\sum_{j=0}^{m}|c_{2j,k}||P_k^{(j)}(t)|^2\right)dt.$$

Wenn der Träger der Funktion $u(x)$ im Äußeren der Einheitskugel liegt, liegen die Träger der Funktionen $P_k(t)$ im Innern der positiven t-Halbachse. Verwendet man die für solche Funktionen gültige Abschätzung [3]

$$\int_0^{\infty}|y^{(j)}(t)|^2dt\geqq\frac{[(2j-1)!!]^2}{4^j}\int_0^{\infty}\frac{|y(t)|^2}{t^{2j}}dt,$$

so folgt aus (15)

$$(16)\qquad \int_{\mathbf{R}^n}(-\Delta)^m u\bar{u}\,dx\geqq\sum_{k=1}^{\infty}\int_0^{\infty}\left(\sum_{j=0}^{m}4^{-j}t^{-2j}[(2j-1)!!]^2|c_{2j,k}|\right)|P_k(t)|^2dt,$$

$$\operatorname{supp}u\in\mathbf{R}^n\setminus K_1,\ \ K_1=\{x\,\big|\,|x|\leqq 1\},\quad u\in C_0^{\infty}(\mathbf{R}^n).$$

Mit

$$\varrho_1(r)=-\left(\sum_{j=0}^{m}\frac{[(2j-1)!!]^2\alpha_j}{4^j\log^{2j}r}\right)r^{-2m}$$

und bei Beachtung der Transformation (11) erhalten wir aus (16)

(17)

$$\int_{\mathbf{R}^n}[(-\Delta)^m u+\varrho_1(r)\,u]\,\bar{u}\,dx\geqq\sum_{k=1}^{\infty}\int_0^{\infty}\left(\sum_{j=0}^{m}4^{-j}t^{-2j}[(2j-1)!!]^2(|c_{2j,k}|-\alpha_j)\right)|P_k(t)|^2dt.$$

Die Zahlen α_j sind mit den Zahlen $|c_{2j,1}|$, $j=0,\dots,m$, identisch. Ist $k\geqq 2$ und damit $k'\geqq 1$, so gilt immer

$$(18)\qquad \alpha_j\leqq|c_{2j,k}|,\quad j=0,\dots,m,\quad k=2,3,\dots.$$

Um (18) zu beweisen, stellen wir fest, daß die Koeffizienten α_j und $|c_{2j,k}|$ Summen von Produkten sind, die aus den Zahlen

$$(19)\qquad 4^{-1}(n-2m)^2,\ 4^{-1}(n-2m+4)^2,\ \dots,\ 4^{-1}(n+2m-8)^2,\ 4^{-1}(n+2m-4)^2,$$

bzw.

$$(20)\qquad 4^{-1}(2k'+n-2m)^2,\ 4^{-1}(2k'+n-2m+4)^2,\ \dots,\ 4^{-1}(2k'+n+2m-4)^2,$$

gebildet werden. Im Falle $n\geqq 2m$ ist (18) offensichtlich. Ist $n<2m$, n ungerade und z. B. $k'=1$, so enthält die Serie (20) zum Teil die Zahlen aus (19), und die restlichen Zahlen aus (20) sind größer als die entsprechenden Zahlen aus (19). Also gilt (18). Entsprechend kann man in den Fällen $k'=2, 3, \dots$ schließen.

Aus (17) folgt jetzt

$$(21)\qquad \int_{\mathbf{R}^n}[(-\Delta)^m u+\varrho_1(r)u]\,\bar{u}\,dx\geqq 0,\quad \operatorname{supp}u\in\mathbf{R}^n\setminus K_1.$$

Ist G ein beliebiges beschränktes Gebiet im $\mathbf{R}^n$, das außerhalb der Kugel K_{r_0} mit dem Radius $r_0>1$ und dem Ursprung als Mittelpunkt liegt, so existiert ein $\eta>0$, so daß $q(x)\geqq\varrho_1(|x|)+\eta$, $x\in G$, gilt. Dann folgt aus (21)

$$\int_G[(-\Delta)^m u+q(x)\,u]\,\bar{u}\,dx\geqq\eta\int_G|u|^2dx,\ \ \eta>0,\ \ u\in C_0^{\infty}(G).$$

Es kann daher keine nichttriviale Funktion $v(x)\in\mathring{W}_2^m(G)$ geben, die Lösung der Differentialgleichung (1) ist. G kann also kein Knotengebiet der Gleichung (1) sein. Wir beweisen jetzt die Behauptung hinsichtlich der Oszillationseigenschaft. Dazu konstruieren wir zu beliebig vorgegebenem $r_2>0$ eine Funktion $u(x)\in C_0^{\infty}(\mathbf{R}^n)$, deren Träger außerhalb der Kugel K_{r_2} liegt und für die

$$(22)\qquad \int_{\mathbf{R}^n}[(-\Delta)^m u+q(x)\,u]\,\bar{u}\,dx<0$$

gilt. Als Testfunktion $u(x)$ wird eine Funktion verwendet, die nur von $|x|=r$ abhängt. Mit

$$\varrho_2(r)=-\left(\sum_{j=0}^{1}\frac{\alpha_j}{4^j\log^{2j}r}\right)r^{-2m}+\frac{\delta(r)}{r^{2m}\log^2 r}$$

folgt dann mittels der Transformation (11)

$$(23)\qquad \int_{\mathbf{R}^n}[(-\Delta)^m u+\varrho_2(r)\,u]\,\bar{u}\,dx=$$

$$=\int_0^{\infty}\left(\sum_{j=0}^{m}\alpha_j|P_1^{(j)}(t)|^2-\sum_{j=0}^{1}4^{-j}t^{-2j}\alpha_j|P_1(t)|^2\right)dt+\int_0^{\infty}t^{-2}\delta(e^t)|P_1(t)|^2dt.$$

Wir setzen jetzt

$$P_1(t)=e^{s/2}\varphi(s),\quad t=e^s,\ \ -\infty<s<\infty,$$

wobei $\varphi(s)$ aus

$$\chi(s)=\begin{cases}1, & \sigma_1\leqq s<\sigma_2,\ \ 1<\sigma_1<\sigma_2-1,\\ 0, & \text{sonst}\end{cases}$$

durch Mittelung mit dem Mittelungsradius $\frac{1}{2}$ entsteht. Dann folgt

$$P_1^{(j)}(t) = e^{(1-2j)/s/2}[a_{j,0}\varphi(s)+a_{j,1}\varphi'(s)+\ldots+a_{j,j}\varphi^{(j)}(s)]$$

und damit

$$|P_1^{(j)}(t)|^2 = e^{(1-2j)s} \sum_{0\leqq\mu,\nu\leqq j} b_{\mu\nu}\varphi^{(\mu)}(s)\varphi^{(\nu)}(s).$$

Also gilt

$$\int_0^\infty |P_1^{(j)}(t)|^2 dt = \int_0^\infty e^{2(1-j)s}\Bigl(\sum_{0\leqq\mu,\nu\leqq j} b_{\mu\nu}\varphi^{(\mu)}(s)\,\varphi^{(\nu)}(s)\Bigr)\,ds = o(1),\ \ \sigma_1,\sigma_2\to\infty, \tag{24}$$

sofern $j\geqq 2$ ist. Weiter ist

$$\int_0^\infty (|P_1'(t)|^2-4^{-1}t^{-2}|P_1(t)|^2)\,dt = \int_0^\infty ([\varphi'(s)]^2+\varphi(s)\varphi'(s))\,ds = C_1<\infty \tag{25}$$

und

$$\int_0^\infty t^{-2}\delta(e^t)|P_1(t)|^2 dt = \int_0^\infty \delta(\exp[e^s])\,\varphi^2(s)\,ds. \tag{26}$$

Werden die Beziehungen (24)—(26) in (23) berücksichtigt, so folgt

$$\int_{\mathrm{R}^n} [(-\varDelta)^m u+\varrho_2(r)\,u]\,\bar{u}\,dx \leqq C_2+\int_0^\infty \delta(\exp[e^s])\,\varphi^2(s)\,ds. \tag{27}$$

Da der Faktor $\varphi^2(s)$ auf $\left[\sigma_1-\frac{1}{2},\sigma_1+\frac{1}{2}\right]$ von $\varphi^2\left(\sigma_1-\frac{1}{2}\right)=0$ bis $\varphi^2\left(\sigma_1+\frac{1}{2}\right)=1$ monoton steigend und auf $\left[\sigma_2-\frac{1}{2},\sigma_2+\frac{1}{2}\right]$ von $\varphi^2\left(\sigma_2-\frac{1}{2}\right)=1$ bis $\varphi^2\left(\sigma_2+\frac{1}{2}\right)=0$ monoton fallend ist, gilt nach dem zweiten Mittelwertsatz der Integralrechnung

$$\int_0^\infty \delta(\exp[e^s])\,\varphi^2(s)\,ds = \int_{\sigma_1'}^{\sigma_2'} \delta(\exp[e^s])\,ds = \int_{r_1'}^{r_2'} \frac{\delta(r)}{r\log r}\,dr,$$

$$\sigma_i-\frac{1}{2} \leqq \sigma_i' \leqq \sigma_i+\frac{1}{2},\ \ \sigma_i' = \log[\log r_i'],\quad i=1,2.$$

Nach Voraussetzung strebt das recht stehende Integral für $\sigma_2\to\infty$ (bei festem σ_1) gegen $-\infty$. Es existiert daher nach (27) eine Funktion $u(x)\in C_0^\infty(\mathrm{R}^n\setminus K_{r_2})$, so daß (22) gilt. Es gelte $u(x)=0$ für $|x|\geqq r_3$, $r_3>r_2$. Die quadratische Form

$$\int_{K_{r_2,r_3}} [(-\varDelta)^m v+q(x)\,v]\bar{v}\,dx,\ \ v\in C_0^\infty(K_{r_2,r_3}),\quad K_{r_2,r_3} = \{x|r_2<|x|<r_3\},$$

ist halbbeschränkt nach unten. Durch sie wird ein selbstadjungierter Operator A im Hilbertraum $L_2(K_{r_2,r_3})$ definiert, dessen Definitionsbereich $D(A)$ im Raum $\mathring{W}_2^m(K_{r_2,r_3})$ liegt. Das Spektrum von A ist diskret, und der kleinste Eigenwert $\lambda_1(r_2,r_3)$ von A ist wegen (22) negativ. Nach dem Variationsprinzip von Courant wird $\lambda_1(r_2,r_3)$ stetig größer, wenn (bei festem r_2) r_3 stetig verkleinert wird. Es gibt einen Wert r^*, $r_2<r^*<r_3$, so daß $\lambda_1(r_2,r^*)=0$ gilt. Für die zugehörige Eigenfunktion $w(x)$ gilt

$$(-\varDelta)^m w+q(x)w = 0,\ \ w\in\mathring{W}_2^m(K_{r_2,r^*})\cap W_2^{2m}(K_{r_2,r^*}).$$

Damit ist nachgewiesen, daß die Gleichung (1) oszilliert, wenn die Bedingung (6) erfüllt ist. Der Satz ist bewiesen.

Die Zahlen

$$\alpha_0 = 4^{-m}\prod_{\nu=0}^{m-1}(n-2m+4\nu)^2,\ n>2m\quad\text{oder}\quad n<2m,\ n\ \text{ungerade}$$

und

$$\tilde{\alpha}_1 = \frac{\alpha_1}{4} = 4^{m-2}[(m-1)!]^2,\quad n=2m,$$

wird man als Kneser-Konstanten bezeichnen.

Spezialfälle: 1. $m\geqq 1$, $n=1$. Für α_0 ergibt sich

$$\alpha_0 = \frac{[(2m-1)!!]^2}{4^m}.$$

Die Bedingungen (5) und (6) für die Gleichung

$$(-1)^m u^{(2m)}+q(x)\,u = 0,\quad 0<x<\infty,$$

sind weniger einschränkend als die entsprechenden von GLAZMAN [3].

2. $m=1$, $n=1$. Die Gleichung

$$-u''+q(x)u = 0,\quad 0<x<\infty,$$

ist nichtoszillatorisch, wenn

$$q(x) > -\frac{1}{4x^2}(1+\log^{-2}x),\quad x>x_0,$$

gilt. Sie ist oszillatorisch, wenn

$$q(x) \leqq -\frac{1}{4x^2}(1+\log^{-2}x)+\frac{\delta(x)}{x^2\log^2 x},\ \ x>x_0,\ \ \int_2^\infty \frac{\delta(x)}{x\log x}\,dx = -\infty,$$

erfüllt ist (vergl. (4)).

3. $m=2$, $n=1$. Die Gleichung

$$u^{(4)}+q(x)u = 0,\quad 0<x<\infty,$$

ist nichtoszillatorisch, wenn

$$q(x) > -\frac{1}{16x^4}(9+10\log^{-2}x+9\log^{-4}x),\quad x>x_0,$$

gilt, und sie ist oszillatorisch, wenn

$$q(x) \leqq -\frac{1}{16x^4}(9+10\log^{-2}x)+\frac{\delta(x)}{x^4\log^2 x}, \quad x > x_0, \quad \int_2^\infty \frac{\delta(x)}{x\log x}\,dx = -\infty,$$

erfüllt ist. Diese Bedingungen verbessern die bekannten Kneser-Typ-Bedingungen von W. LEIGHTON und Z. NEHARI [6] für die vorliegende Gleichung.

4. $m=1$, $n\geqq 3$. $-\Delta u+q(x)u=0$, $x\in\mathbf{R}^n$, ist nichtoszillatorisch, wenn

$$q(x) > -\frac{1}{4|x|^2}([n-2]^2+\log^{-2}|x|), \quad |x| > r_0,$$

gilt. Im Falle

$$q(x) \leqq -\frac{1}{4|x|^2}([n-2]^2+\log^{-2}|x|)+\frac{\delta(|x|)}{|x|^2\log^2|x|}, \quad |x| > r_0, \quad \int_2^\infty \frac{\delta(r)}{r\log r}\,dr = -\infty,$$

ist die Schrödingergleichung oszillatorisch. Diese Bedingungen sind weniger einschränkend als die entsprechenden von GLAZMAN [3]. $m=1$, $n=2$: Jetzt ist $\alpha_0=0$, und die Bedingungen heißen

$$q(x) > -\frac{1}{4|x|^2\log^2|x|}, \quad |x| > r_0, \tag{28}$$

bzw.

$$q(x) \leqq -\frac{1}{4|x|^2\log^2|x|}+\frac{\delta(|x|)}{|x|^2\log^2|x|}, \quad |x| > r_0, \quad \int_2^\infty \frac{\delta(r)}{r\log r}\,dr = -\infty.$$

Die Bedingung (28) ist (mit Einschluß der Gleichheit) von ALLEGRETTO [1] angegeben worden.

5. $m=2$, $n=1$ oder $n\geqq 3$. Die Bedingungen heißen

$$q(x) > -\frac{1}{|x|^4}\left(\frac{(n-4)^2n^2}{16}+\frac{n^2-4n+8}{8\log^2|x|}+\frac{6}{16\log^4|x|}\right), \quad |x| > r_0, \tag{29}$$

bzw.

$$q(x) \leqq -\frac{1}{|x|^4}\left(\frac{(n-4)^2n^2}{16}+\frac{n^2-4n+8}{8\log^2|x|}\right)+\frac{\delta(|x|)}{|x|^4\log^2|x|},$$

$$|x| > r_0, \quad \int_2^\infty \frac{\delta(r)}{r\log r}\,dr = -\infty.$$

6. $m=3$, $n=1,3$ oder $n\geqq 5$. Die Bedingungen heißen

$$q(x) > -\frac{1}{|x|^6}\left(\frac{(n-6)^2(n-2)^2(n+2)^2}{64}+\frac{3n^4-24n^3+72n^2-96n+304}{64\log^2|x|}+\right.$$

$$\left.+\frac{9(3n^2-12n+44)}{64\log^4|x|}+\frac{225}{64\log^6|x|}\right), \quad |x| > r_0,$$

bzw.

$$q(x) \leqq -\frac{1}{|x|^6}\left(\frac{(n-6)^2(n-2)^2(n+2)^2}{64}+\frac{3n^4-24n^3+72n^2-96n+304}{64\log^2|x|}+\frac{\delta(|x|)}{|x|^6\log^2|x|}\right.,$$

$$|x| > r_0; \quad \int_2^\infty \frac{\delta(r)}{r\log r}\,dr = -\infty.$$

Wir diskutieren jetzt den Fall $n<2m$, n gerade. Zunächst sei $n=2m-4l$, $l=0, 1, 2, \dots$. Dann ist $\alpha_0=0$. Problematisch ist der Beweis der Beziehungen (18) für $j=1, \dots, m$; $k=2, 3, \dots$. Wir vermuten, daß sie allgemein richtig sind, so daß der erste Teil des Satzes 1 auch im vorliegenden Fall richtig bleibt. Was die Oszillation betrifft, so gilt

SATZ 2. *Es sei $n=2m-4l$, $l=0, 1, 2, \dots$. Die Differentialgleichung*

$$(-\Delta)^m u+q(x)u = 0, \quad x\in\mathbf{R}^n,$$

ist oszillatorisch, wenn eine Funktion $\delta(r)$ mit der Eigenschaft

$$\int_2^\infty \frac{\delta(r)}{r\log r}\,dr = -\infty$$

und eine Zahl $r_0>0$ existieren, so daß

$$q(x) \leqq -\frac{\tilde{\alpha}_1}{|x|^{2m}\log^2|x|}+\frac{\delta(|x|)}{|x|^{2m}\log^2|x|}, \quad |x| > r_0,$$

$$\tilde{\alpha}_1 = \frac{\alpha_1}{4} = 4^{m-2}\prod_{\substack{\nu=0\\ \nu\neq l}}^{m-1}(\nu-l)^2,$$

gilt.

Der Beweis des Satzes 2 kann dem Beweis des zweiten Teils von Satz 1 unmittelbar entnommen werden.

Für konkrete Zahlen m und n können die Beziehungen (18) durch Rechnung bestätigt werden. So ergibt sich z. B. folgendes.

Im Falle $m=3$, $n=2$ ist die Gleichung

$$(-\Delta)^3 u+q(x)u = 0, \quad x\in\mathbf{R}^2,$$

nichtoszillatorisch oder oszillatorisch, je nachdem ob

$$q(x) > -\frac{1}{|x|^6\log^2|x|}\left(4+\frac{9}{2\log^2|x|}+\frac{225}{64\log^4|x|}\right), \quad |x| > r_0,$$

oder

$$q(x) \leqq -\frac{4}{|x|^6\log^2|x|}+\frac{\delta(|x|)}{|x|^6\log^2|x|}, \quad |x| > r_0, \quad \int_2^\infty \frac{\delta(r)}{r\log r}\,dr = -\infty,$$

gilt.

Wir kommen schließlich zu dem Fall $n=2m-2-4l$, $l=0, 1, \dots$. Anstelle der Zahlen α_j verwenden wir jetzt die Zahlen β_j, die als die Koeffizienten des Polynoms

$$\xi^m+\beta_{m-1}\xi^{m-1}+\dots+\beta_1\xi+\beta_0 =$$

$$= \left[\xi+\left(\frac{n-2m+2}{2}\right)^2\right]\left[\xi+\left(\frac{n-2m+6}{2}\right)^2\right]\dots\left[\xi+\left(\frac{n+2m-2}{2}\right)^2\right], \quad \beta_m = 1,\ \beta_0 = 0,$$

definiert sind. Mit

$$\varrho_1(r) = -\left(\sum_{j=0}^{m}\frac{[(2j-1)!!]^2\beta_j}{4^j\log^{2j} r}\right)r^{-2m}$$

folgt die der Ungleichung (17) entsprechende Ungleichung

(30)
$$\int_{\mathbf{R}^n}[(-\Delta)^m u+\varrho_1(r)\,u]\,\bar{u}\,dx \geqq \sum_{k=1}^{\infty}\int_0^{\infty}\left(\sum_{j=0}^{m}4^{-j}t^{-2j}[(2j-1)!!]^2(|c_{2j,k}|-\beta_j)\right)|P_k(t)|^2\,dt.$$

Dabei ist

(31) $$\beta_0 = 0,\ |c_{0,1}| = \prod_{\nu=0}^{m-1}[1+2(l-\nu)]^2 > 0, \quad n = 2m-2-4l.$$

Für alle $k\geqq 2$, für die in (9) $k'=1$ ist, gilt

$$|c_{2j,k}| = \beta_j, \quad j = 0, \dots, m; \quad k' = 1.$$

Wenn die Ungleichungen

(32) $$|c_{2j,k}| \geqq \beta_j, \quad j = 0, \dots, m, \quad \text{für} \quad k' \geqq 2,$$

gelten, so folgt aus (30)

(33)
$$\int_{\mathbf{R}^n}[(-\Delta)^m u+\varrho_1(r)\,u]\,\bar{u}\,dx \geqq \int_0^{\infty}\left(|c_{0,1}|+\sum_{j=1}^{m}4^{-j}t^{-2j}[(2j-1)!!]^2(|c_{2j,1}|-\beta_j)\right)|P_1(t)|^2\,dt.$$

Aus (33) folgt, daß wegen (31) ein $t_1>0$ existiert und entsprechend ein $r_1=e^{t_1}$, so daß

$$\int_{\mathbf{R}^n}[(-\Delta)^m u+\varrho_1(r)\,u]\,\bar{u}\,dx \geqq 0, \quad |x| > r_1,$$

gilt. Nun schließt man wie im Beweis des Satzes 1 und erhält den Satz: *Sofern die Beziehungen* (32) *richtig sind, ist die Gleichung*

$$(-\Delta)^m u+q(x)u = 0, \quad x\in\mathbf{R}^n, \quad n = 2m-2-4l, \quad l = 0, 1, \dots,$$

nichtoszillatorisch, wenn die Bedingung

$$q(x) > -\left(\sum_{j=1}^{m}\frac{[(2j-1)!!]^2\beta_j}{4^j\log^{2j}|x|}\right)|x|^{-2m}, \quad |x| > r_1,$$

erfüllt ist.

Was die Oszillation betrifft, so gilt der gleiche Sachverhalt wie im Satz 2.

SATZ 3. *Es sei* $n=2m-2-4l$, $l=0, 1, \dots$. *Die Gleichung*

$$(-\Delta)^m u+q(x)u = 0, \quad x\in\mathbf{R}^n,$$

ist oszillatorisch, wenn eine Funktion $\delta(r)$ *mit der Eigenschaft*

$$\int_2^{\infty}\frac{\delta(r)}{r\log r}\,dr = -\infty$$

und eine Zahl $r_0>0$ *existieren, so daß*

$$q(x) \leqq -\frac{\tilde{\beta}_1}{|x|^{2m}\log^2|x|}+\frac{\delta(|x|)}{|x|^{2m}\log^2|x|}, \quad |x| > r_0,$$

$$\tilde{\beta}_1 = \frac{\beta_1}{4} = 4^{m-2}\prod_{\substack{\nu=0\\ \nu\neq l}}^{m-1}(\nu-l)^2,$$

gilt.

BEWEIS. Wir verwenden als Testfunktion

$$v(x) = S_2(\vartheta)\,R_2(r), \quad R_2(r) = 0, \quad 0 \leqq r \leqq r_0, \quad r_0 > 1.$$

Die Kugelflächenfunktion $S_2(\vartheta)$ gehört zum Eigenwert $s_2=-(n-1)$, $(k'=1)$. Es gilt

$$\int_{\mathbf{R}^n}(-\Delta)^m v\cdot\bar{v}\,dx = \int_0^{\infty}\left(\sum_{j=1}^{m}|c_{2j,2}|\,|P_2^{(j)}(t)|^2\right)dt,$$

und mit

$$\varrho_2(r) = -\frac{\beta_1}{4r^{2m}\log^2 r}+\frac{\delta(r)}{r^{2m}\log^2 r}; \quad \beta_1 = |c_{2,2}|,$$

folgt

$$\int_{\mathbf{R}^n}[(-\Delta)^m v+\varrho_2(r)\,v]\,\bar{v}\,dx =$$

$$= \int_0^{\infty}\left(\sum_{j=1}^{m}|c_{2j,2}|\,|P_2^{(j)}(t)|^2-4^{-1}t^{-2}|c_{2,2}|\,|P_2(t)|^2\right)dt+\int_0^{\infty}t^{-2}\delta(e^t)|P_2(t)|^2\,dt.$$

Setzt man wieder

$$P_2(t) = e^{s/2}\varphi(s), \quad t=e^s, \quad -\infty< s <\infty,$$

so erhält man wie im Beweis von Satz 1

$$\int_0^{\infty}\left(\sum_{j=1}^{m}|c_{2j,2}|\,|P_2^{(j)}(t)|^2-4^{-1}t^{-2}|c_{2,2}|\,|P_2(t)|^2\right)dt \leqq C.$$

Der Summand

$$\int_0^{\infty}t^{-2}\delta(e^t)|P_2(t)|^2\,dt$$

strebt für $\sigma_2 \to \infty$ gegen $-\infty$, so daß wieder

$$\int_{\mathbf{R}^n} [(-\Delta)^m v + q(x)\, v]\, \bar{v}\, dx < 0$$

erreicht wird. Damit folgt die Behauptung des Satzes wie oben im Beweis von Satz 1.

Für konkrete Zahlen m und n können die Beziehungen (32) bestätigt und folgende Aussagen gewonnen werden.

Spezialfälle: 1. $m=2$, $n=2$. *Die Gleichung*

$$\Delta^2 u + q(x) u = 0, \quad x \in \mathbf{R}^2,$$

ist nichtoszillatorisch, wenn

$$q(x) > -\frac{1}{|x|^4 \log^2 |x|}\left(1 + \frac{9}{16} \log^{-2} |x|\right), \quad |x| > r_0,$$

gilt. Im Falle

$$q(x) \leqq -\frac{1}{|x|^4 \log^2 |x|} + \frac{\delta(|x|)}{|x|^4 \log^2 |x|}, \quad |x| > r_0, \quad \int_2^\infty \frac{\delta(r)}{r \log r}\, dr = -\infty,$$

ist die Gleichung oszillatorisch.

2. $m=3$, $n=4$. *Die Bedingungen heißen*

$$q(x) > -\frac{1}{|x|^6 \log^2 |x|}\left(16 + \frac{45}{4 \log^2 |x|} + \frac{225}{64 \log^4 |x|}\right), \quad |x| > r_0,$$

bzw.

$$q(x) \leqq -\frac{16}{|x|^6 \log^2 |x|} + \frac{\delta(|x|)}{|x|^6 \log^2 |x|}, \quad |x| > r_0, \quad \int_2^\infty \frac{\delta(r)}{r \log r}\, dr = -\infty.$$

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ASYMPTOTIC ESTIMATES FOR QUASI-HERMITE—FEJÉR INTERPOLATION

By

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1. Introduction

About twenty years ago, PAUL SZÁSZ [4] introduced the interpolation polynomials

$$(1) \qquad Q_{2n+1}(f; x) = \frac{f(-1)(1-x)U_n(x)^2}{2(n+1)^2} + \frac{f(+1)(1+x)U_n(x)^2}{2(n+1)^2} + \sum_{k=1}^{n} f(x_k)(1-x^2)\frac{(1-xx_k)}{(n+1)^2}\left(\frac{U_n(x)}{x-x_k}\right)^2$$

where the points $x_k = \cos\frac{k\pi}{n+1}$ for $k=1, 2, \ldots, n$, are the zeros of the Chebyshev polynomial of the second kind

$$U_n(x) = \frac{\sin(n+1)\theta}{\sin\theta}, \quad -1 \leqq x = \cos\theta \leqq 1.$$

Szász termed these polynomials the *quasi-Hermite—Fejér interpolation polynomials* based on the zeros of $U_n(x)$. Theorem II of Szász's paper implies the following result:

THEOREM 1 (P. Szász). *If* $f \in C([-1, 1])$ *then*

$$\lim_{n\to\infty} \|Q_{2n+1}(f) - f\|_\infty = 0.$$

Eight years later, R. B. SAXENA [3] estimated the rate of convergence of these polynomials. Theorem 2 of Saxena's paper may be stated as follows:

THEOREM 2 (R. B. Saxena). *If* $f \in C([-1, 1])$, *then for* $n = 2, 3, 4, \ldots$,

$$\|Q_{2n+1}(f) - f\| \leqq 5\omega\left(f; \frac{\log n}{n}\right)$$

where $\omega(f; \delta)$ *is the modulus of continuity of the function* f.

It is not difficult to refine Saxena's result to:

THEOREM 3. *If* $f \in C([-1, 1])$, $-1 \leqq x \leqq 1$ *and* $n > 1$, *then*

$$|Q_{2n+1}(f; x) - f(x)| \leqq \frac{C}{n} \sum_{k=1}^{n} \omega\left(f; \frac{\sqrt{1-x^2}}{k}\right)$$

where C *is an absolute constant.*

As the methods used to prove Theorem 3 are contained in the papers by SAXENA [3] and by PRASAD and VARMA [2], there is no need to prove Theorem 3 here.

This paper arose from our interest in the constants 5 and C in the preceding two theorems. We shall obtain the best asymptotic constant in the case where f belongs to a Lipschitz class of functions. Specifically, we shall confine our attention to the Lipschitz class Lip 1, where $f \in$ Lip 1 means that

$$|f(x)-f(y)| \leqq |x-y| \quad \text{for all } x, y \text{ in } [-1, 1].$$

It follows immediately that $f \in$ Lip 1 if and only if $\omega(f;\ \delta) \leqq \delta$ for all $\delta \geqq 0$.

We then obtain the following corollary of Theorem 2.

COROLLARY 1. *If* $f \in$ Lip 1, *then for* $n=2, 3, 4, \ldots$,

$$\|Q_{2n+1}(f)-f\|_\infty \leqq 5\frac{\log n}{n}.$$

The aim of this paper is to estimate the quantity

$$\Delta_n(x) := \sup\{|Q_{2n+1}(f; x)-f(x)| : f \in \text{Lip } 1\} \quad \text{for} \quad -1 \leqq x \leqq 1.$$

Our main result may be stated as follows:

THEOREM 4. *If* $-1 \leqq x \leqq 1$, *then*

$$\Delta_n(x) = \sqrt{1-x^2}\,|U_n(x)| \left[\frac{2}{\pi}(1-x^2)|U_n(x)|\frac{\log n}{n} + O\left(\frac{1}{n}\right)\right] \quad as \quad n \to \infty.$$

Note that Theorem 4 reflects the fact that the interpolation polynomials $Q_{2n+1}(f; x)$ and the function $f(x)$ agree at the zeros of $(1-x^2)U_n(x)$.

An immediate corollary of this theorem is

COROLLARY 2.

$$\sup\{\|Q_{2n+1}(f)-f\|_\infty : f \in \text{Lip } 1\} = \frac{2}{\pi}\frac{\log n}{n} + O\left(\frac{1}{n}\right), \quad as \quad n \to \infty.$$

2. Proof of Theorem 4

To prove Theorem 4, we shall begin by rewriting equation (1) as follows:

$$Q_{2n+1}(f; x) = f(-1)\,r_{n+1}(x) + f(1)\,r_0(x) + \sum_{k=1}^{n} f(x_k)\,r_k(x).$$

Then by an argument first proposed by S. M. NIKOLSKII [1],

$$\Delta_n(x) = (1+x)\,r_{n+1}(x) + (1-x)\,r_0(x) + \sum_{k=1}^{n} |x-x_k|\,r_k(x). \tag{2}$$

The first step in estimating $\Delta_n(x)$ is to note that

$$(1+x)\,r_{n+1}(x) + (1-x)\,r_0(x) = \frac{(1-x^2)\,U_n(x)^2}{(n+1)^2} = \tag{3}$$

$$= \frac{\sqrt{1-x^2}\,|U_n(x)|}{n+1}\,\frac{|\sin(n+1)\,\theta|}{n+1} = O\left(\frac{\sqrt{1-x^2}\,|U_n(x)|}{n+1}\right),$$

where $x = \cos\theta$.

Let $x \in [-1,\ 1]$. We may assume that x is not a node of interpolation for, in that case, $\Delta_n(x) = 0$ and there is nothing to prove. Choose the index j such that

$$|x-x_j| \leqq |x-x_k| \quad \text{for} \quad k = 1, 2, \ldots, n.$$

Clearly $j = j(x, n)$. If there are two such values of j, then choose either one. Then we have

$$\sum_{k=1}^{n} |x-x_k|\,r_k(x) = \sum_{k=1}^{n} W_k(x) = \tag{4}$$

$$= \sum_{k=1}^{j-2} W_k(x) + W_{j-1}(x) + W_j(x) + W_{j+1}(x) + \sum_{k=j+2}^{n} W_k(x).$$

Of course, if $j = 1, 2, n-1$ or n, then some of these terms will not appear.

Now

$$\frac{1-xx_j}{\sqrt{1-x_j^2}} = \frac{1-x_j^2+x_j^2-xx_j}{\sqrt{1-x_j^2}} \leqq \sqrt{1-x_j^2} + \frac{|x_j|\,|x-x_j|}{\sqrt{1-x^2}} \leqq 1 + \frac{\pi}{2}. \tag{5}$$

Furthermore,

$$W_j(x) = |x-x_j|\,r_j(x) = \frac{|x-x_j|(1-x^2)(1-xx_j)}{(n+1)^2}\left(\frac{U_n(x)}{x-x_j}\right)^2.$$

From SZÁSZ [4, p. 426], we have

$$\frac{1-x^2}{1-x_j^2}\,[l_j(x)]^2 < 2 \quad \text{for all} \quad x \in [-1, 1] \tag{6}$$

where $l_j(x)$ is the jth fundamental polynomial of Lagrange interpolation corresponding to the nodes $x_1, x_2, \ldots, x_n$.

It follows that

$$W_j(x) = \frac{\sqrt{1-x^2}\,|U_n(x)|}{n+1} \cdot \frac{1-xx_j}{\sqrt{1-x_j^2}} \cdot \sqrt{\frac{1-x^2}{1-x_j^2}}\,|l_j(x)| \leqq \tag{7}$$

$$\leqq \frac{\sqrt{1-x^2}\,|U_n(x)|}{n+1}\left(1+\frac{\pi}{2}\right)\sqrt{2} = O\left(\frac{\sqrt{1-x^2}\,|U_n(x)|}{n+1}\right)$$

by (5) and (6).

Similarly we can show

$$W_{j-1}(x) = O\left(\frac{\sqrt{1-x^2}\,|U_n(x)|}{n+1}\right) \tag{8}$$

and

$$W_{j+1}(x) = O\left(\frac{\sqrt{1-x^2}|U_n(x)|}{n+1}\right). \tag{9}$$

Returning to (4), we find that for $k=1, 2, \ldots, j-2$,

$$W_k(x) = \frac{(1-x^2)(1-xx_k)}{(n+1)^2}\cdot\frac{U_n(x)^2}{x_k-x} = \frac{(1-x^2)^2U_n(x)^2}{(n+1)^2(x_k-x)} - \frac{x(1-x^2)U_n(x)^2}{(n+1)^2} = \tag{10}$$

$$= \frac{(1-x^2)^2U_n(x)^2}{(n+1)^2(x_k-x)} + O\left(\frac{\sqrt{1-x^2}|U_n(x)|}{(n+1)^2}\right) = V_k(x) + O\left(\frac{\sqrt{1-x^2}|U_n(x)|}{(n+1)^2}\right),$$

say. Furthermore,

$$\sum_{k=1}^{j-2} V_k(x) = \frac{\sin^2\theta\sin^2(n+1)\theta}{(n+1)^2}\sum_{k=1}^{j-2}\frac{1}{\cos\theta_k-\cos\theta} =$$

$$= \frac{\sin^2\theta\sin^2(n+1)\theta}{\pi(n+1)}\left[\int_0^\theta\left(\frac{1}{\cos y-\cos\theta}-\frac{1}{(\theta-y)\sin\theta}\right)dy+o(1)\right]+$$

$$+\frac{\sin\theta\sin^2(n+1)\theta}{(n+1)^2}\sum_{k=1}^{j-2}\frac{1}{\theta-\theta_k} =$$

$$= \frac{\sin\theta\sin^2(n+1)\theta}{\pi(n+1)}\left[\log\left(\frac{2\sin\theta}{\theta}\right)+o(1)\right]+\frac{\sin\theta\sin^2(n+1)\theta}{(n+1)^2}\sum_{k=1}^{j-2}\frac{1}{\theta-\theta_k}$$

where $x=\cos\theta$. Thus

$$\sum_{k=1}^{j-2} V_k(x) = \frac{\sin\theta\sin^2(n+1)\theta}{(n+1)^2}\sum_{k=1}^{j-2}\frac{1}{\theta-\theta_k}+O\left(\frac{\sqrt{1-x^2}|U_n(x)|}{n+1}\right) = \tag{11}$$

$$= \frac{\sin\theta\sin^2(n+1)\theta\log j}{\pi(n+1)}+O\left(\frac{\sqrt{1-x^2}|U_n(x)|}{n+1}\right).$$

From (10) and (11), we have

$$\sum_{k=1}^{j-2} W_k(x) = \frac{\sin\theta\sin^2(n+1)\theta\log j}{\pi(n+1)}+O\left(\frac{\sqrt{1-x^2}|U_n(x)|}{n+1}\right).$$

Similarly we can show

$$\sum_{k=j+2}^{n} W_k(x) = \frac{\sin\theta\sin^2(n+1)\theta\log(n-j)}{\pi(n+1)}+O\left(\frac{\sqrt{1-x^2}|U_n(x)|}{n+1}\right).$$

It follows that

(12)

$$\left(\sum_{k=1}^{j-2}+\sum_{k=j+2}^{n}\right)W_k(x) = \frac{\sin\theta\sin^2(n+1)\theta\log j(n-j)}{\pi(n+1)}+O\left(\frac{\sqrt{1-x^2}|U_n(x)|}{n+1}\right).$$

Now

$$j(n-j) = (n+1)^2\left(\frac{\theta}{\pi}+O\left(\frac{1}{n}\right)\right)\left(\frac{\pi-\theta}{\pi}+O\left(\frac{1}{n}\right)\right)$$

and hence

$$\log j(n-j) = 2\log(n+1)+\log\theta+\log(\pi-\theta)+O(1). \tag{13}$$

Furthermore,

$$\sin(n+1)\theta\cdot\log\theta = O(1) \quad \text{for} \quad 0\leqq\theta\leqq\pi \tag{14}$$

and

$$\sin(n+1)\theta\cdot\log(\pi-\theta) = O(1) \quad \text{for} \quad 0\leqq\theta\leqq\pi. \tag{15}$$

Substituting (13), (14) and (15) into (12), we have

(16)

$$\left(\sum_{k=1}^{j-2}+\sum_{k=j+2}^{n}\right)W_k(x) = \frac{2\sin\theta\log(n+1)\sin^2(n+1)\theta}{\pi(n+1)}+O\left(\frac{\sqrt{1-x^2}|U_n(x)|}{n+1}\right) =$$

$$= \sqrt{1-x^2}|U_n(x)|\left[\frac{2}{\pi}(1-x^2)|U_n(x)|\frac{\log n}{n}+O\left(\frac{1}{n}\right)\right].$$

Equations (2), (3), (4), (7), (8), (9) and (16) prove Theorem 4.

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ON ADDITIVE REPRESENTATIONS OF INTEGERS. II

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1. Throughout this paper, we write $e^{2\pi i\alpha}=e(\alpha)$. $c_1, c_2, \dots$ denote positive absolute constants.

In Part I of this paper (see [4]), my aim was to analyse and generalize the most important applications of the Hardy—Littlewood method. In fact, I proved the following theorems:

THEOREM 1. *Let $P, Q, N, k, a_1, a_2, \dots, a_y$ be positive integers such that*

$$2P < Q \leqq N, \tag{1}$$

$$PQ \leqq N, \tag{2}$$

$$a_1 < a_2 < \dots < a_y \leqq N \tag{3}$$

and

$$k \geqq 3. \tag{4}$$

Let $d_1, d_2, \dots, d_y$ be non-negative real numbers such that $d_i \neq 0$ holds for at least one i with $1 \leqq i \leqq y$ and let us put

$$\max_{1 \leqq j \leqq y} d_j = \Delta,$$

$$D(u, q, h) = \sum_{\substack{a_i \leqq u \\ a_i \equiv h \pmod{q}}} d_i, \quad D(u) = D(u, 1, 0) = \sum_{a_i \leqq u} d_i,$$

$$S_u(\alpha) = \sum_{a_i \leqq u} d_i e(a_i \alpha), \quad S(\alpha) = S_N(\alpha) = \sum_{i=1}^{y} d_i e(a_i \alpha),$$

and for $j=1, 2, \dots,$ denote the number of solutions of the equation

$$a_{x_1}+a_{x_2}+\dots+a_{x_j} = a_{z_1}+a_{z_2}+\dots+a_{z_j}, \quad 1 \leqq x_1, \dots, x_j, \; z_1, \dots, z_j \leqq y$$

by R_{2j}.

Let us assume that the following conditions hold:

(i) There exist real numbers K, A and a real function $F(u)$ (defined for $0<u<+\infty$) such that

$$|D(u)-F(u)| < K \quad for \quad 1 \leqq u \leqq N, \tag{5}$$

$$K \geqq 1, \tag{6}$$

$F(u)$ has continuous second derivative in $(0, N]$,

$$F(1) = 1, \tag{7}$$

$$F'(N) > 0, \tag{8}$$

$$F''(u) \leqq 0 \quad \text{for} \quad 1 \leqq u \leqq N \tag{9}$$

and

$$uF'(u) \leqq AF(u) \quad \text{for} \quad 1 \leqq u \leqq N. \tag{10}$$

(ii) *There exist positive real numbers* $L_1, L_2, \dots, L_P$, *and for* $1 \leqq q \leqq P$, $0 \leqq h \leqq q$, *there exist real numbers* $m(q, h)$ *such that*

$$\sum_{h=0}^{q-1} |D(u, q, h) - m(q, h) D(u)| < L_q \quad \text{for} \quad 1 \leqq q \leqq P \quad \text{and} \quad 0 \leqq u \leqq N. \tag{11}$$

Write

$$T(q, b) = \sum_{h=0}^{q-1} m(q, h)\, e(hb/q) \quad (\text{for } 1 \leqq q \leqq P \text{ and all } b),$$

$$T_q = \max_{\substack{0 \leqq b < q \\ (b,q)=1}} |T(q, b)| \quad (\text{for } 1 \leqq q \leqq P)$$

and

$$T = T^{(N)} = \sum_{q=1}^{P} \sum_{\substack{0 \leqq b < q \\ (b,q)=1}} \big(T(q, b)\big)^k e(-Nb/q).$$

(iii) *There exists a real number* H *such that if* b, q *are integers satisfying* $P < q \leqq Q$, $(b, q) = 1$, *and* $0 < u \leqq N$, *then we have*

$$|S_u(b/q)| < H. \tag{12}$$

If all the conditions above hold then we have

$$\left| \sum_{\substack{1 \leqq x_1, \dots, x_k \leqq y \\ a_{x_1} + \dots + a_{x_k} = N}} \prod_{i=1}^{k} d_{x_i} - T \sum_{\substack{1 \leqq n_1, \dots, n_k \leqq N \\ n_1 + \dots + n_k = N}} \prod_{j=1}^{k} F'(n_j) \right| < \tag{13}$$

$$< \min_{1 \leqq j < k/2} \left(8 \frac{HN}{PQ}\right)^{k-2j} \Delta^{2j} R_{2j} + \sum_{q=1}^{P} 2^{3k+1} k (A+5)^k q (L_q + T_q K) \Big\{ L_q^{k-1} N (qQ)^{-2} +$$

$$+ T_q^{k-1} \Big((K + F(N))^{k-1} N^{-1} + K^{k-1} N^k (qQ)^{-(k+1)} + N \int_{qQ}^{N} x^{-3} (F(x))^{k-1} dx \Big) \Big\} +$$

$$+ \sum_{q=1}^{P} 2(2A+1)^k q T_q^k \int_{2}^{qQ} x^{-2} \big(F(x)\big)\, dx.$$

THEOREM 2. *Assume that all the conditions* (1)—(12) *in Theorem* 1 *hold, furthermore, we have*

$$|T| (F'(N))^k \binom{N-1}{k-1} > \min_{1 \leqq j < k/2} \left(8 \frac{HN}{PQ}\right)^{k-2j} \Delta^{2j} R_{2j} + \tag{14}$$

$$+ \sum_{q=1}^{P} 2^{3k+1} k (A+5)^k q (L_q + T_q K) \Big\{ L_q^{k-1} N (qQ)^{-2} +$$

$$+ T_q^{k-1} \Big((K + F(N))^{k-1} N^{-1} + K^{k-1} N^k (qQ)^{-(k+1)} + N \int_{qQ}^{N} x^{-3} (F(x))^{k-1} dx \Big) \Big\} +$$

$$+ \sum_{q=1}^{P} 2(2A+1)^k q T_q^k \int_{2}^{qQ} x^{-2} \big(F(x)\big)^k dx.$$

Then N can be represented in the form

$$a_{x_1} + a_{x_2} + \dots + a_{x_k} = N. \tag{15}$$

Furthermore, I showed in [4] that these theorems cover the Goldbach—Waring problem, by proving that for fixed s, the numbers p^s form an asymptotic basis of order less than $5s^2 \log s$ for large s. (In the proof of this fact, I used also Vinogradov's and Hua's deep estimates in order to deduce an estimate of type (12) and to obtain an upper bound for R_{2j}.)

One might like to seek for possibly general theorems which can be used in order to show that the conditions in Theorems 1 and 2 hold; this is the aim of this paper. In fact, in Section 2 we prove theorems which can be used to estimate the numbers T_q and T (in other words, we consider the so called "singular series"), while in Section 3 we seek for conditions which imply an estimate of type (12). In Section 4, we combine the results of Sections 2 and 3 with Theorems 1 and 2, in order to consider the simplest case $k=3$. Finally, in Section 5 we show that the theorem proved in Section 4 covers Goldbach's problem (concerning the representability of large odd integers as the sum of three primes).

2. It is reasonable to assume that the numbers $m(q, h)$ defined in the condition (ii) of Theorem 1 are such that for $(q_1, q_2) = 1$ and all h we have

$$m(q_1 q_2, h) = m(q_1, h) \cdot m(q_2, h). \tag{16}$$

In fact, this assumption says that for distinct moduli q_1, q_2 (satisfying $(q_1, q_2) = 1$), the distributions among the residue classes modulo q_1 and q_2, respectively, are independent; this holds in all the known applications.

Throughout this section, we estimate the numbers T_q and T assuming that (16) holds. In particular, we show that (16) implies that the estimate of the numbers T_q can be reduced to the special case $q = p^\alpha$ (Theorems 3 and 4), furthermore, we obtain lower bounds for $|T|$ (Theorems 5 and 6).

LEMMA 1. *Let q_1, q_2 be positive integers such that $(q_1, q_2) = 1$. For $q = q_1$, q_2 and $q_1 q_2$, respectively, let $\dots$, $m(q, -2)$, $m(q, -1)$, $m(q, 0)$, $m(q, 1)$, $m(q, 2)$, $\dots$ be any complex numbers such that for all h,*

$$m(q, h) = m(q, h+q) \tag{17}$$

(with $q=q_1, q_2$ and q_1q_2) and (16) hold, and put

$$T(q, b) = \sum_{h=0}^{q-1} m(q, h)\, e(hb/q) \tag{18}$$

(for $q=q_1, q_2, q_1q_2$, and for all b). Let q_1^*, q_2^* denote integers satisfying

$$q_1 q_1^* \equiv 1 \pmod{q_2} \quad \textit{and} \quad q_2 q_2^* \equiv 1 \pmod{q_1}.$$

Then we have

$$T(q_1 q_2, b) = T(q_1, bq_2^*)T(q_2, bq_1^*)$$

for all b.

PROOF. By $(q_1, q_2)=1$, for any integer h there exist uniquely determined integers μ, ν such that $0 \leqq \mu \leqq q_2-1$, $0 \leqq \nu \leqq q_1-1$ and

$$\mu q_1 + \nu q_2 \equiv h \pmod{q_1 q_2}.$$

Thus we have

$$T(q_1q_2, b) = \sum_{h=0}^{q_1q_2-1} m(q_1q_2, h)\, e(hb/q_1q_2) =$$

$$= \sum_{\mu=0}^{q_2-1} \sum_{\nu=0}^{q_1-1} m(q_1q_2, \mu q_1+\nu q_2)\, e((\mu q_1+\nu q_2)\, b/q_1q_2) =$$

$$= \sum_{\mu=0}^{q_2-1} \sum_{\nu=0}^{q_1-1} m(q_1, \mu q_1+\nu q_2)\, m(q_2, \mu q_1+\nu q_2)\, e(\mu b/q_2)\, e(\nu b/q_1) =$$

$$= \sum_{\mu=0}^{q_2-1} \sum_{\nu=0}^{q_1-1} m(q_1, \nu q_2)\, m(q_2, \mu q_1)\, e(\mu b/q_2)\, e(\nu b/q_1) =$$

$$= \left(\sum_{\nu=0}^{q_1-1} m(q_1, \nu q_2)\, e(\nu b/q_1)\right)\left(\sum_{\mu=0}^{q_2-1} m(q_2, \mu q_1)\, e(\mu b/q_2)\right) =$$

$$= \left(\sum_{\gamma=0}^{q_1-1} m(q_1, \gamma)\, e(\gamma q_2^* b/q_1)\right)\left(\sum_{\delta=0}^{q_2-1} m(q_2, \delta)\, e(\delta q_1^* b/q_2)\right) = T(q_1, q_2^* b)\, T(q_2, q_1^* b).$$

THEOREM 3. *Let us use the same notations as in Lemma 1 and assume that the same conditions hold (in particular, we assume that $(q_1, q_2)=1$ and (16) hold). Put*

$$T_q = \max_{\substack{0 \leqq b < q \\ (b, q)=1}} |T(q, b)| \tag{19}$$

(for $q=q_1, q_2$ and q_1q_2). Then we have $T_{q_1q_2} \leqq T_{q_1} T_{q_2}$.

PROOF. By Lemma 1, we have

$$T_{q_1q_2} = \max_{\substack{0 \leqq b < q_1q_2 \\ (b, q_1q_2)=1}} |T(q_1q_2, b)| =$$

$$= \max_{\substack{0 \leqq b < q_1q_2 \\ (b, q_1q_2)=1}} |T(q_1, bq_2^*)|\,|T(q_2, bq_1^*)| \leqq \max_{\substack{0 \leqq b < q_1q_2 \\ (b, q_1q_2)=1}} T_{q_1} T_{q_2} = T_{q_1} T_{q_2}.$$

THEOREM 4. *Let q be a positive integer. For $d|q$, let $\dots, m(d, -2), m(d, -1), m(d, 0), m(d, 1), m(d, 2), \dots$ be any complex numbers such that for all h, we have $m(d, h)=m(d, h+d)$, and for $(q_1, q_2)=1$, $q_1q_2|q$ (and all h) also (16) holds. For $d|q$, define $T(d, b)$ and T_d by (18) and (19) (with d in place of q). Then we have*

$$T_q \leqq \prod_{p^\alpha \| q} T_{p^\alpha}.$$

(Here $\prod_{p^\alpha \| q}$ denotes that that p^α runs over the prime powers p^α satisfying $p^\alpha | q$, $p^{\alpha+1} \nmid q$.)

PROOF. This follows trivially from Theorem 3.

LEMMA 2. *Let us use the same notations as in Lemma 1 and assume that the same conditions hold (in particular, we assume that $(q_1, q_2)=1$ and (16) hold). Furthermore, let k be a positive integer, N an integer, and write*

$$B(q, N) = \sum_{\substack{0 \leqq b < q \\ (b, q)=1}} (T(q, b))^k e(-Nb/q) \tag{20}$$

(for $q=q_1, q_2, q_1q_2$). Then we have

$$B(q_1 q_2, N) = B(q_1, N)B(q_2, N).$$

PROOF. If $(b, q)=1$ then by $(q_1, q_2)=1$, there exist uniquely determined integers μ, ν such that

$$b \equiv \mu q_1 + \nu q_2 \pmod{q_1 q_2},$$

$$0 \leqq \mu < q_2, \quad (\mu, q_2) = 1 \quad \text{and} \quad 0 \leqq \nu < q_1, \quad (\nu, q_1) = 1.$$

Thus with respect to Lemma 1 (and since $T(q, b_1)=T(q, b_2)$ for $b_1 \equiv b_2 \pmod{q}$), we have

$$B(q_1q_2, N) = \sum_{\substack{0 \leqq b < q_1q_2 \\ (b, q_1q_2)=1}} (T(q_1q_2, b))^k e(-Nb/q_1q_2) =$$

$$= \sum_{\substack{0 \leqq \mu < q_2 \\ (\mu, q_2)=1}} \sum_{\substack{0 \leqq \nu < q_1 \\ (\nu, q_1)=1}} (T(q_1q_2, \mu q_1+\nu q_2))^k e(-N(\mu q_1+\nu q_2)/q_1q_2) =$$

$$= \sum_{\substack{0 \leqq \mu < q_2 \\ (\mu, q_2)=1}} \sum_{\substack{0 \leqq \nu < q_1 \\ (\nu, q_1)=1}} (T(q_1, (\mu q_1+\nu q_2)\, q_2^*))^k (T(q_2, (\mu q_1+\nu q_2) q_1^*))^k e(-N\mu/q_2)\, e(-N\nu/q_1) =$$

$$= \sum_{\substack{0 \leqq \mu < q_2 \\ (\mu, q_2)=1}} \sum_{\substack{0 \leqq \nu < q_1 \\ (\nu, q_1)=1}} (T(q_1, \nu))^k (T(q_2, \mu))^k e(-N\mu/q_2)\, e(-N\nu/q_1) =$$

$$= \sum_{\substack{0 \leqq \nu < q_1 \\ (\nu, q_1)=1}} (T(q_1, \nu))^k e(-N\nu/q_1) \sum_{\substack{0 \leqq \mu < q_2 \\ (\mu, q_2)=1}} (T(q_2, \mu))^k e(-N\mu/q_2) = B(q_1, N)B(q_2, N).$$

LEMMA 3. *Let q, k be positive integers, N an integer. Let us use the same notations as in Theorem 4 and assume that the same conditions hold. Define $B(q, N)$ by (20). Then we have*

$$|B(q, N)| \leqq \varphi(q)\, T_q^k \leqq \prod_{p^\alpha \| q} \varphi(p^\alpha) T_{p^\alpha}^k.$$

PROOF. By Theorem 4, we have

$$|B(q,N)| = \Big|\sum_{\substack{0\leqq b<q\\(b,q)=1}} \big(T(q,b)\big)^k e(-Nb/q)\Big| \leqq$$

$$\leqq \sum_{\substack{0\leqq b<q\\(b,q)=1}} |T(q,b)|^k \leqq \sum_{\substack{0\leqq b<q\\(b,q)=1}} T_q^k = \varphi(q)\,T_q^k \leqq \prod_{p^\alpha\|q} \varphi(p^\alpha) T_{p^\alpha}^k.$$

LEMMA 4. *Let k be a positive integer, N an integer. For $q=1,2,\ldots$, let $\ldots, m(q,-2)$, $m(q,-1)$, $m(q,0)$, $m(q,1)$, $m(q,2)$, $\ldots$ be any complex numbers such that for all h, q and for $(q_1,q_2)=1$, (16) and (17) hold, furthermore, there exist q, h with $m(q,h)\neq 0$. For all q and b, define $T(q,b)$, T_q and $B(q,N)$ by (18), (19) and (20). Assume that*

$$\sum_{q=1}^{+\infty} \prod_{p^\alpha\|q} \varphi(p^\alpha) T_{p^\alpha}^k < +\infty \tag{21}$$

(i.e., this infinite series of positive terms converges). Then for all prime number p, the infinite series

$$A(p,N) = 1+\sum_{\alpha=1}^{+\infty} B(p^\alpha, N)$$

is absolutely convergent, also both the infinite series

$$U^{(N)} = \sum_{q=1}^{+\infty} B(q,N) \tag{22}$$

and the infinite product

$$\prod_p A(p,N)$$

are absolutely convergent and we have

$$U^{(N)} = \sum_{q=1}^{+\infty} B(q,N) = \prod_p A(p,N).$$

PROOF. By Lemma 3, we have

$$|B(q,N)| \leqq \prod_{p^\alpha\|q} \varphi(p^\alpha) T_{p^\alpha}^k,$$

thus the infinite series in (21) majorizes each of the infinite series $U^{(N)}$, $A(p,N)$ and

$$\sum_p |A(p,N)-1| = \sum_p \Big|\sum_{\alpha=1}^{+\infty} B(p^\alpha,N)\Big| \leqq \sum_p \sum_{\alpha=1}^{+\infty} |B(p^\alpha,N)|.$$

Thus in fact, both infinite series $U^{(N)}$, $A(p,N)$ and the infinite product $\prod_p A(p,N)$ are absolutely convergent, and by Lemma 2, we have

$$\prod_p A(p,N) = \prod_p \Big(1+\sum_{\alpha=1}^{+\infty} B(p^\alpha,N)\Big) =$$

$$= 1+\sum_{q=2}^{+\infty} \Big(\prod_{p^\alpha\|q} B(p^\alpha,N)\Big) = 1+\sum_{q=2}^{+\infty} B(q,N) = \sum_{q=1}^{+\infty} B(q,N) = U^{(N)}$$

(note that $m(q,h)\not\equiv 0$ and (16) imply that $B(1,N)=1$) which completes the proof of Lemma 4.

THEOREM 5. *Let us use the same notations as in Lemma 4 and assume that the same conditions hold (in particular, $m(q,h)\not\equiv 0$, (17) and for $(q_1,q_2)=1$, (16) hold). Let E, ε and ω be positive numbers such that*

$$\sum_{q=1}^{+\infty} \prod_{p^\alpha\|q} \varphi(p^\alpha) T_{p^\alpha}^k < E, \tag{23}$$

$$|A(p,N)| \geqq \varepsilon \quad \text{for all} \quad p<\omega \tag{24}$$

and

$$\sum_{\alpha=1}^{+\infty} \varphi(p^\alpha) T_{p^\alpha}^k < \frac{1}{2} \quad \text{for all} \quad p\geqq\omega. \tag{25}$$

Then there exist positive numbers $\eta=\eta(E,\varepsilon,\omega)$ and $P_0=P_0(E,\varepsilon,\omega)$ such that

$$|T| = |T^{(N)}| = \Big|\sum_{q=1}^{P} B(q,N)\Big| > \eta \tag{26}$$

for $P\geqq P_0$ and all N. (Note that η is independent of N and P!)

PROOF. By Lemma 4, the infinite series (22) is absolutely convergent and we have

$$|U^{(N)}| = \Big|\sum_{q=1}^{+\infty} B(q,N)\Big| = \Big|\prod_p A(p,N)\Big| = \prod_{p<\omega} |A(p,N)| \prod_{p\geqq\omega} |A(p,N)| \geqq \tag{27}$$

$$\geqq \prod_{p<\omega} \varepsilon \prod_{p\geqq\omega} \Big|1+\sum_{\alpha=1}^{+\infty} B(p^\alpha,N)\Big| \geqq \varepsilon^\omega \prod_{p\geqq\omega} \Big|1+\sum_{\alpha=1}^{+\infty} B(p^\alpha,N)\Big|.$$

By (25) and Lemma 3, for all $p\geqq\omega$ we have

$$\Big|\sum_{\alpha=1}^{+\infty} B(p^\alpha,N)\Big| \leqq \sum_{\alpha=1}^{+\infty} |B(p^\alpha,N)| \leqq \sum_{\alpha=1}^{+\infty} \varphi(p^\alpha) T_{p^\alpha}^k < \frac{1}{2} \quad (\text{for } p\geqq\omega). \tag{28}$$

Furthermore, for $0\leqq x<1/2$ we have

$$e^{-2x} = 1+\sum_{n=1}^{+\infty} (-1)^n \frac{(2x)^n}{n!} \leqq 1-2x+\frac{2x^2}{2!} = 1-2x+2x^2 \leqq 1-2x+2x\cdot\frac{1}{2} = 1-x.$$

Thus (27) and (28) yield that

$$|U^{(N)}| \geqq \varepsilon^\omega \prod_{p\geqq\omega} \Big(1-\Big|\sum_{\alpha=1}^{+\infty} B(p^\alpha,N)\Big|\Big) \geqq \varepsilon^\omega \prod_{p\geqq\omega} \Big(1-\sum_{\alpha=1}^{+\infty} |B(p^\alpha,N)|\Big) \geqq \tag{29}$$

$$\geqq \varepsilon^\omega \prod_{p\geqq\omega} e^{-2\sum_{\alpha=1}^{+\infty} |B(p^\alpha,N)|} \geqq \varepsilon^\omega e^{-2\sum_{p\geqq\omega}\sum_{\alpha=1}^{+\infty} \varphi(p^\alpha) T_{p^\alpha}^k} > \varepsilon^\omega e^{-2E}.$$

By Lemma 4, (23) implies that the infinite series (22) is absolutely convergent; thus there exists a positive integer $P_0=P_0(E, \varepsilon, \omega)$ such that we have

$$\sum_{q>P_0} |B(q, N)| < \frac{\varepsilon^{\omega} e^{-2E}}{2}. \tag{30}$$

(29) and (30) yield that for $P \geqq P_0$,

$$|T| = |T^{(N)}| = \left|\sum_{q=1}^{P} B(q, N)\right| \geqq \left|\sum_{q=1}^{+\infty} B(q, N)\right| - \left|\sum_{q>P} B(q, N)\right| \geqq$$

$$\geqq |U^{(N)}| - \sum_{q>P} |B(q, N)| > \varepsilon^{\omega} e^{-2E} - \sum_{q>P_0} |B(q, N)| > \frac{\varepsilon^{\omega} e^{-2E}}{2}.$$

Thus (26) holds with $\eta=\varepsilon^{\omega} e^{-2E}/2$ which completes the proof of Theorem 5.

In some important special cases (like Goldbach's problem and the Goldbach—Waring problem), also the following theorem can be used:

THEOREM 6. *Let us use the same notations as in Lemma* 4 *and assume that the same conditions hold (in particular* , $m(q, h) \not\equiv 0$, (17) *and for* $(q_1, q_2)=1$, (16) *hold). Assume that* (21) *holds and let* E, ω $(\geqq 1)$ *be positive numbers satisfying* (23) *and* (25). *Assume that for all* $p<\omega$, *there exists a positive integer* α_p *such that for* $\alpha \geqq \alpha_p$ *and all* M *we have*

$$B(p^{\alpha}, M) = 0. \tag{31}$$

Then there exist positive numbers η *and* P_0 *such that* (26) *holds for* $P \geqq P_0$ *and all* N *satisfying*

$$A(p, N) \neq 0 \quad \text{for all} \quad p < \omega. \tag{32}$$

PROOF. (31) implies that for

$$N_1 \equiv N_2 \pmod{p^{\alpha_p}},$$

we have

$$A(p, N_1) = 1 + \sum_{\alpha=1}^{+\infty} B(p^{\alpha}, N_1) = 1 + \sum_{\alpha=1}^{\alpha_p} B(p^{\alpha}, N_1) =$$

$$= 1 + \sum_{\alpha=1}^{\alpha_p} B(p^{\alpha}, N_2) = 1 + \sum_{\alpha=1}^{+\infty} B(p^{\alpha}, N_2) = A(p, N_2).$$

Thus for fixed p and $N=1, 2, \dots, A(p, N)$ assumes only finitely many distinct values and all these values occur for $1 \leqq N \leqq p^{\alpha_p}$. Writing

$$\varepsilon = \min_{\substack{p,N \\ p<\omega, 1 \leqq N \leqq p^{\alpha_p} \\ A(p,N) \neq 0}} |A(p, N)|,$$

(24) in Theorem 5 holds by (32), thus Theorem 5 can be applied with this number ε and this completes the proof of Theorem 6.

3. In this section, we seek for conditions which yield upper bound for the modulus of sums of the form $\sum_{h=0}^{q-1} z_h e(hb/q)$ (where $(b, q)=1$). Namely, by using theorems

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of this type, we may prove inequalities of the form (12) (and in some cases, also upper bounds for the numbers T_q can be obtained in this way).

THEOREM 7. *Let* q *be a positive integer satisfying*

$$q > 1, \tag{33}$$

$\dots, z_{-2}, z_{-1}, z_0, z_1, z_2, \dots$ *any complex numbers such that* $z_{n+q}=z_n$ *for all* n, δ *any real number satisfying*

$$0 < \delta \leqq \frac{1}{6}. \tag{34}$$

Put

$$\sum_{j=0}^{q-1} z_j = Z \tag{35}$$

and

$$\max_{\substack{0 \leqq h < q \\ (h,q)=1}} \sum_{t=0}^{q-1} \left| \sum_{t-\delta q < l \leqq t} z_{hl} - \delta Z \right| = \Gamma. \tag{36}$$

If b *is any integer satisfying* $(b, q)=1$ *then we have*

$$\left|\sum_{h=0}^{q-1} z_h e(hb/q)\right| < 2\Gamma/\delta q.$$

PROOF. For $h=0, 1, \dots, q-1$, define the integer h' by

$$h' \equiv hb \pmod{q}, \quad 0 \leqq h' \leqq q-1. \tag{37}$$

By $(b, q)=1$, there exists a uniquely determined integer b^* for which

$$bb^* \equiv 1 \pmod{q}, \quad 0 \leqq b^* \leqq q-1.$$

Then obviously, $(b^*, q)=1$. (37) implies that $h \equiv h'b^*$ (mod q). Furthermore, if h runs over the numbers $0, 1, \dots, q-1$, then again by $(b, q)=1$, also h' runs over these numbers. Thus we have

$$\sum_{h=0}^{q-1} z_h e(hb/q) = \sum_{h'=0}^{q-1} z_{h'b^*} e(h'/q).$$

Hence (with respect to (33) and (36))

$$\left|\sum_{0 \leqq j < \delta q} e(j/q) \sum_{h=0}^{q-1} z_h e(hb/q)\right| = \left|\sum_{0 \leqq j < \delta q} e(j/q) \sum_{h'=0}^{q-1} z_{h'b^*} e(h'/q)\right| = \tag{38}$$

$$= \left|\sum_{t=0}^{q-1} \Big(\sum_{t-\delta q < l \leqq t} z_{lb^*}\Big) e(t/q)\right| = \left|\sum_{t=0}^{q-1} \delta Z e(t/q) + \sum_{t=0}^{q-1} \Big(\sum_{t-\delta q < l \leqq t} z_{lb^*} - \delta Z\Big) e(t/q)\right| =$$

$$= \left|\sum_{t=0}^{q-1} \Big(\sum_{t-\delta q < l \leqq t} z_{lb^*} - \delta Z\Big) e(t/q)\right| \leqq \sum_{t=0}^{q-1} \left|\sum_{t-\delta q < l \leqq t} z_{lb^*} - \delta Z\right| \leqq \Gamma.$$

By (34), we have

$$(39)\quad \Big|\sum_{0\leqq j<\delta q} e(j/q)\Big| \geqq \Big|\sum_{0\leqq j<\delta q} \mathrm{Re}\, e(j/q)\Big| = \Big|\sum_{0\leqq j<\delta q} \cos 2\pi j/q\Big| = \sum_{0\leqq j<\delta q} \cos 2\pi j/q >$$

$$> \sum_{0\leqq j<\delta q} \cos 2\pi\delta q/q \geqq \sum_{0\leqq j<\delta q} \cos \pi/3 = \frac{1}{2}\sum_{0\leqq j<\delta q} 1 > \delta q/2.$$

(38) and (39) yield that

$$\Big|\sum_{h=0}^{q-1} z_h e(hb/q)\Big| \leqq \frac{\Gamma}{\Big|\sum\limits_{0\leqq j<\delta q} e(j/q)\Big|} < \frac{\Gamma}{\delta q/2} = 2\Gamma/\delta q$$

which completes the proof of Theorem 7.

In order to estimate the parameter Γ defined in Theorem 7, the following theorem can be used:

THEOREM 8. *Let q, h, t be integers satisfying $q>2$ and $(q, h)=1, \dots, z_{-2}, z_{-1}, z_0, z_1, z_2, \dots$ any complex numbers such that*

$$(40)\qquad z_{n+q} = z_n \quad \textit{for all } n$$

and

$$(41)\qquad z_m = 0 \quad \textit{for all } m \textit{ satisfying } \ (q, m) > 1.$$

Define Z by (35). Then for all $\delta>0$, we have

$$\Big|\sum_{t-\delta q<l\leqq t} z_{hl} - \delta Z\Big| \leqq c_1 q^{-1/2} \log\log q \left(\log q \sum_{\chi\neq\chi_0} \Big|\sum_{j=0}^{q-1} \chi(j) z_j\Big| + |Z|\right).$$

(Here and in what follows, $\sum_{\chi}$ denotes summation over all the characters χ modulo q, while in $\sum_{\chi\neq\chi_0}$, χ runs over all the characters χ modulo q, different from the principal character χ_0.)

Note that Theorem 8 can be used also in the more general case when (41) does not hold; in fact, in this case, we have

$$\sum_{t-\delta q<l\leqq t} z_{hl} - \delta Z = \sum_{d|q} \Big(\sum_{\substack{t/d-\delta q/d<l\leqq t/d \\ (l, q/d)=1}} z_{hdl} - \delta \sum_{\substack{1\leqq j\leqq q/d \\ (j, q/d)=1}} z_{dj} \Big)$$

and here each term can be estimated by using Theorem 8.

PROOF. By (40) and (41), we have

$$z_n = \frac{1}{\varphi(q)} \sum_{\chi} \bar\chi(n) \sum_{j=0}^{q-1} \chi(j) z_j.$$

Thus

$$(42)\qquad \sum_{t-\delta q<l\leqq t} z_{hl} = \sum_{t-\delta q<l\leqq t} \frac{1}{\varphi(q)} \sum_{\chi} \bar\chi(hl) \sum_{j=0}^{q-1} \chi(j) z_j =$$

$$= \frac{1}{\varphi(q)} \sum_{\chi} \bar\chi(h) \sum_{t-\delta q<l\leqq t} \bar\chi(l) \sum_{j=0}^{q-1} \chi(j) z_j = \frac{1}{\varphi(q)} \bar\chi_0(h) \sum_{t-\delta q<l\leqq t} \bar\chi_0(l) \sum_{j=0}^{q-1} \chi_0(j) z_j +$$

$$+ \frac{1}{\varphi(q)} \sum_{\chi\neq\chi_0} \bar\chi(h) \sum_{t-\delta q<l\leqq t} \bar\chi(l) \sum_{j=0}^{q-1} \chi(j) z_j =$$

$$= \frac{Z}{\varphi(q)} \sum_{\substack{t-\delta q<l\leqq t \\ (l,q)=1}} 1 + \frac{1}{\varphi(q)} \sum_{\chi\neq\chi_0} \bar\chi(h) \sum_{t-\delta q<l\leqq t} \bar\chi(l) \sum_{j=0}^{q-1} \chi(j) z_j.$$

Now we are going to show that

$$(43)\qquad \Big|\sum_{\substack{t-\delta q<l\leqq t \\ (l,q)=1}} 1 - \delta\varphi(q)\Big| < 2\sqrt{q}$$

(for all $q\geqq 2$). Let $q = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r}$ where $p_1, p_2, \dots, p_r$ are distinct prime numbers, $\alpha_1, \alpha_2, \dots, \alpha_r$ are positive integers. Then we have

$$\sum_{\substack{t-\delta q<l\leqq t \\ (l,q)=1}} 1 = \sum_{t-\delta q<l\leqq t} 1 + \sum_{j=1}^{r} (-1)^j \sum_{1\leqq i_1<\dots<i_j\leqq r} \sum_{\substack{t-\delta q<l\leqq t \\ p_{i_1}\dots p_{i_j}|l}} 1$$

and

$$\delta\varphi(q) = \delta q \prod_{i=1}^{r} \left(1 - \frac{1}{p_i}\right) = \delta q + \sum_{j=1}^{r} (-1)^j \sum_{1\leqq i_1 \dots i_j\leqq r} \frac{\delta q}{p_{i_1}\dots p_{i_j}}$$

thus denoting the number of the positive divisors of an integer n by $d(n)$,

$$\Big|\sum_{\substack{t-\delta q<l\leqq t \\ (l,q)=1}} 1 - \delta\varphi(q)\Big| =$$

$$= \left|\Big(\sum_{t-\delta q<l\leqq t} 1 - \delta q\Big) + \sum_{j=1}^{r} (-1)^j \sum_{1\leqq i_1<\dots<i_j\leqq r} \Big(\sum_{\substack{t-\delta q<l\leqq t \\ p_{i_1}\dots p_{i_j}|l}} 1 - \frac{\delta q}{p_{i_1}\dots p_{i_j}}\Big)\right| \leqq$$

$$\leqq \Big|\sum_{t-\delta q<l\leqq t} 1 - \delta q\Big| + \sum_{j=1}^{r} \sum_{1\leqq i_1<\dots<i_j\leqq r} \Big|\sum_{\substack{t-\delta q<l\leqq t \\ p_{i_1}\dots p_{i_j}|l}} 1 - \frac{\delta q}{p_{i_1}\dots p_{i_j}}\Big| <$$

$$< 1 + \sum_{j=1}^{r} \sum_{1\leqq i_1<\dots<i_j\leqq r} 1 = 1 + \sum_{j=1}^{r} \binom{r}{j} = 2^r =$$

$$= d(p_1 p_2 \dots p_r) \leqq d(p_1^{\alpha_1} p_2^{\alpha_2} \dots p_r^{\alpha_r}) = d(q) \leqq 2\sqrt{q}$$

(since it is well-known that $d(q)\leqq 2\sqrt{q}$ for any positive integer q) which proves (43).

(42) and (43) yield that

$$\left|\sum_{t-\delta q<l\le t} z_{hl}-\delta Z\right| \le \tag{44}$$

$$\le \left|\sum_{t-\delta q<l\le t} z_{hl}-\frac{Z}{\varphi(q)}\sum_{\substack{t-\delta q<l\le t\\(l,q)=1}} 1\right|+\frac{|Z|}{\varphi(q)}\left|\sum_{\substack{t-\delta q<l\le t\\(l,q)=1}} 1-\delta\varphi(q)\right| \le$$

$$\le \frac{1}{\varphi(q)}\left|\sum_{\chi\ne\chi_0}\bar\chi(h)\sum_{t-\delta q<l\le t}\bar\chi(l)\sum_{j=0}^{q-1}\chi(j)z_j\right|+\frac{2\sqrt{q}}{\varphi(q)}|Z| \le$$

$$\le \frac{1}{\varphi(q)}\sum_{\chi\ne\chi_0}\left|\sum_{t-\delta q<l\le t}\bar\chi(l)\right|\left|\sum_{j=0}^{q-1}\chi(j)z_j\right|+\frac{2\sqrt{q}}{\varphi(q)}|Z|.$$

By the Pólya—Vinogradov inequality (see [2] and [5]), we have

$$\left|\sum_{U<n\le V}\chi(n)\right|< c_2\sqrt{q}\log q \tag{45}$$

for all q, U, V and any character $\chi\ne\chi_0$ modulo q. Furthermore, it is well-known that for $n\ge 3$ we have

$$\varphi(n)>c_3\frac{n}{\log\log n} \tag{46}$$

(see e.g. [3], p. 24). By using (45) and (46), we obtain from (44) that

$$\left|\sum_{t-\delta q<l\le t} z_{hl}-\delta Z\right| \le \frac{1}{\varphi(q)}\sum_{\chi\ne\chi_0} c_2\sqrt{q}\log q\left|\sum_{j=0}^{q-1}\chi(j)z_j\right|+\frac{2\sqrt{q}}{\varphi(q)}|Z| \le$$

$$\le c_4\frac{\sqrt{q}}{\varphi(q)}\left(\log q\sum_{\chi\ne\chi_0}\left|\sum_{j=0}^{q-1}\chi(j)z_j\right|+|Z|\right) \le$$

$$\le c_5 q^{-1/2}\log\log q\left(\log q\sum_{\chi\ne\chi_0}\left|\sum_{j=0}^{q-1}\chi(j)z_j\right|+|Z|\right)$$

which completes the proof of Theorem 8.

4. As also Part I of this paper shows, the critical points in the applications of Theorems 1 and 2 are the estimates of H and R_{2j}. In the special case $k=3$, the estimate of R_{2j} is trivial, and Theorem 2 can be simplified also from other points of view. Thus it is worth to consider this special case separately.

THEOREM 9. *Let $a_1<a_2<\dots$, $N_1<N_2<\dots$ be infinite sequences of positive integers, $d_1<d_2<\dots$ an infinite sequence of positive real numbers. Let us put*

$$\delta_j=\min_{\substack{i\\a_i\le N_j}} d_i,\quad \Delta_j=\max_{\substack{i\\a_i\le N_j}} d_i,$$

$$D(u,q,h)=\sum_{\substack{a_i\le u\\a_i\equiv h\,(\mathrm{mod}\,q)}} d_i,\quad D(u)=D(u,1,0)=\sum_{a_i\le u} d_i$$

and

$$S_u(\alpha)=\sum_{a_i\le u} d_i e(a_i\alpha).$$

For $j=1,2,\dots$, let P_j, Q_j be positive integers such that

$$2<2P_j<Q_j\le N_j/2, \tag{47}$$

$$P_jQ_j\le N_j \tag{48}$$

and

$$\lim_{j\to+\infty} P_j=+\infty, \tag{49}$$

and assume that the following conditions hold:

(i) *There exist real numbers $K_1, K_2, \dots$ and a real function $F(u)$ (defined for $0<u<+\infty$) such that*

$$|D(u)-F(u)|<K_j\quad \textit{for}\quad j=1,2,\dots,\quad 1\le u\le N_j, \tag{50}$$

$$K_j\ge 1\quad\textit{for}\quad j=1,2,\dots,$$

$F(u)$ has continuous second derivative for $u>0$,

$$F(1)=1,\quad F'(u)>0\quad\textit{for}\quad u\ge 1,\quad F''(u)\le 0\quad\textit{for}\quad u\ge 1$$

and

$$\frac{2}{3}F(u)\le uF'(u)\le F(u)\quad\text{for}\quad u\ge 1. \tag{51}$$

(ii) *There exist real numbers $M_1, M_2, \dots$ and for $q=1,2,\dots$, $h=0,\pm1,\pm2,\dots$ there exist non-negative real numbers $m(q,h)$ such that*

$$m(q_1q_2,h)=m(q_1,h)\,m(q_2,h)\quad (\textit{for all}\ (q_1,q_2)=1), \tag{52}$$

$$\sum_{h=0}^{q-1} m(q,h)\le 1 \tag{53}$$

and

$$\sum_{h=0}^{q-1}|D(u,q,h)-m(q,h)D(u)|<M_j\quad\textit{for}\quad 1\le q\le P_j\quad\textit{and}\quad 0\le u<N_j. \tag{54}$$

Write

$$T(q,b)=\sum_{h=0}^{q-1} m(q,h)\,e(hb/q),\quad T_q=\max_{0\le b<q}|T(q,b)|,$$

$$B(q,N)=\sum_{\substack{0\le b<q\\(b,q)=1}}\big(T(q,b)\big)^3 e(-Nb/q),$$

$$T^{(N_j)}=\sum_{q=1}^{P_j} B(q,N_j)=\sum_{q=1}^{P_j}\sum_{\substack{0\le b<q\\(b,q)=1}}\big(T(q,b)\big)^3 e(-N_jb/q)$$

and

$$A(p, N) = 1+\sum_{\alpha=1}^{+\infty} B(p^\alpha, N)$$

(provided that this infinite series is convergent).

(iii) *For $j\to+\infty$, the numbers K_j, M_j, T_q defined above satisfy the assumptions*

(55)
$$(M_j+K_j)(M_j^2N_j^2Q_j^{-2}\log P_j+N_j^2(F(Q_j))^2Q_j^{-2}P_j^2\log N_jQ_j^{-1}+K_j^2N_j^4Q_j^{-4}) = o(F^3(N_j))$$

and

$$\sum_{q=1}^{P_j} T_q^3q^3 \leqq \sum_{q=1}^{P_j}\prod_{p^\alpha\|q} T_{p^\alpha}^3p^{3\alpha} = o\left(\frac{Q_j}{N_j}\left(\frac{F(N_j)}{F(Q_j)}\right)^3\right). \tag{56}$$

(iv) *We have*

$$\sum_{q=1}^{+\infty}\prod_{p^\alpha\|q}\varphi(p^\alpha)T_{p^\alpha}^3 < +\infty. \tag{57}$$

Furthermore, if ω is a positive number such that

$$\sum_{\alpha=1}^{+\infty}\varphi(p^\alpha)T_{p^\alpha}^3 < \frac{1}{2} \quad \text{for all} \quad p \geqq \omega \tag{58}$$

then we have

$$\lim_{j\to+\infty}\inf|A(p, N_j)| > 0 \quad \text{for all} \quad p < \omega. \tag{59}$$

(v) *For $j=1, 2, \ldots$, there exists a real number H_j such that if b, q, u are integers satisfying $P_j<q\leqq Q_j$, $(b, q)=1$ and $0<u\leqq N_j$, then*

$$|S_u(b/q)| < H_j, \tag{60}$$

and we have

$$H_j = o\left(\frac{P_jQ_j}{N_j^2\Delta_j^2}F^2(N_j)\right). \tag{61}$$

If all the assumptions above hold, then for $j>j_0$, N_j can be represented in the form

$$a_x+a_y+a_z = N_j. \tag{62}$$

Furthermore, the same conclusion holds if the conditions (iv) *and* (v), *respectively, are replaced by the following conditions:*

(iv′) *Assume that* (57) *holds and let ω satisfy* (58). *Then for all $p<\omega$, there exists a positive integer α_p such that for $\alpha\geqq\alpha_p$ and all j we have $B(p^\alpha, N_j)=0$. Furthermore, for all j and $p<\omega$ we have $A(p, N_j)\neq 0$.*

(v′) *For $j=1, 2, \ldots$, there exists a real number H_j such that if q is an integer satisfying $P_j<q\leqq Q_j$, then there exists a real number δ such that $0<\delta\leqq 1/6$, and for any integer u satisfying $0<u\leqq N_j$, we have*

$$\max_{\substack{0\leqq h<q\\(h,q)=1}}\sum_{t=0}^{q-1}\left|\sum_{t-\delta q<l\leqq t} D(u, q, hl)-\delta D(u)\right| \leqq \frac{\delta q}{2}H_j; \tag{63}$$

furthermore, this number H_j satisfies (61),

(v″) *For $j=1, 2, \ldots$, there exists a real number G_j such that if q is an integer satisfying $P_j<q\leqq Q_j$, then for $0<u\leqq N_j$ we have*

(64)
$$\min\left\{q^{-1/2}\log\log q\left(\log q\sum_{\chi\neq\chi_0}\left|\sum_{k=0}^{q-1}\chi(k)D(u, q, k)\right|+D(u)\right)+\sum_{\substack{a_l\leqq u\\(a_l,q)>1}} d_l, D(u)\right\} < G_j;$$

furthermore, this number G_j satisfies

$$G_j = o\left(\frac{P_jQ_j}{N_j^2\Delta_j^2}F^2(N_j)\right). \tag{65}$$

Proof. We are going to show that our assumptions imply that for $j>j_0$ Theorem 2 can be applied with P_j, Q_j, N_j in place of P, Q, N, and with $k=3$.

In fact, define $y(=y(j))$ by

$$y = \sum_{\substack{l\\a_l\leqq N_j}} 1.$$

Then with respect to (47) and (48), conditions (1)—(4) in Theorem 1 hold trivially. For $j>j_0$, $d_l\neq 0$ holds for at least one i with $1\leqq i\leqq y$.

The condition (i) in Theorem 9 implies trivially that also condition (i) in Theorem 1 holds (with $K=K_j$).

(54) yields that (11) (in Theorem 1) holds with

$$L_1 = L_2 = \ldots = L_{P_j} = M_j. \tag{66}$$

(12) (with $H=H_j$) holds by (60).

Finally, we have to show that also (14) holds for sufficiently large j.

Writing

$$\frac{1}{2}\min_{p<\omega}\lim_{j\to+\infty}\inf A(p, N_j) = \varepsilon,$$

we have $\varepsilon>0$ by (59) and for $j>j_0$, (24) in Theorem 5 holds. By (57) and (58), also (23) (with a number E independent of j) and (24) hold. Thus Theorem 5 can be applied and we obtain for sufficiently large j that

$$|T| = |T^{(N_j)}| > \eta(>0) \tag{67}$$

(where η is independent of j).

Thus with respect to (51), for sufficiently large j the left hand side of (14) can be estimated in the following way:

(68)
$$|T|(F'(N))^k\binom{N-1}{k-1} = |T^{(N_j)}|(F'(N_j))^3\binom{N_j-1}{2} >$$
$$> \eta\left(\frac{2}{3}N_j^{-1}F(N_j)\right)^3\frac{N_j^2}{2} > \frac{\eta}{7}(F(N_j))^3N_j^{-1}.$$

On the other hand, with respect to (50), the first term on the right hand side of (14) can be estimated in the following way:

$$\min_{1\leqq j<k/2}\left(8\frac{HN}{PQ}\right)^{k-2j}\Delta^{2j}R_{2j}=8\frac{H_jN_j}{P_jQ_j}\Delta_j^2R_2= \tag{69}$$

$$=8\frac{H_jN_j}{P_jQ_j}\Delta_j^2\sum_{a_x=a_z\leqq N_j}1=8\frac{H_jN_j}{P_jQ_j}\Delta_j^2\sum_{a_i\leqq N_j}1\leqq$$

$$\leqq 8\frac{H_jN_j}{P_jQ_j}\Delta_j^2\sum_{a_i\leqq N_j}\frac{d_i}{\delta_j}=8\frac{H_jN_j}{P_jQ_j}\frac{\Delta_j^2}{\delta_j}D(N_j)<8\frac{H_jN_j}{P_jQ_j}\frac{\Delta_j^2}{\delta_j}(F(N_j)+K_j).$$

(55) implies that

$$K_j\cdot K_j^2N_j^4Q_j^{-4}=K_j^3N_j^4Q_j^{-4}=o\big(F^3(N_j)\big)$$

thus by (47),

$$K_j=o\big(F(N_j)(Q_j/N_j)^4\big)=o\big(F(N_j)\big). \tag{70}$$

(69) and (70) yield that for $j>j_0$, we have

$$\min_{1\leqq j<k/2}\left(8\frac{HN}{PQ}\right)^{k-2j}\Delta^{2j}R_{2j}<8\frac{H_jN_j}{P_jQ_j}\frac{\Delta_j^2}{\delta_j}2F(N_j)=o\big((F(N_j))^3N_j^{-1}\big). \tag{71}$$

Furthermore, by (66) and (70), for $j\to+\infty$ the second term on the right hand side of (14) can be estimated in the following way:

$$\sum_{q=1}^{P}2^{3k+1}k(A+5)^kq(L_q+T_qK)\Big\{L_q^{k-1}N(qQ)^{-2}+ \tag{72}$$

$$+T_q^{k-1}\Big(\big(K+F(N)\big)^{k-1}N^{-1}+K^{k-1}N^k(qQ)^{-(k+1)}+N\int_{qQ}^{N}x^{-3}\big(F(x)\big)^{k-1}dx\Big)\Big\}=$$

$$=O\left(\sum_{q=1}^{P_j}q(M_j+T_qK_j)\Big\{M_j^2N_j(qQ_j)^{-2}+T_q^2\Big(\big(F(N_j)\big)^2N_j^{-1}+K_j^2N_j^3(qQ_j)^{-4}+\right.$$

$$\left.+N_j\int_{qQ_j}^{N_j}x^{-3}\big(F(x)\big)^2dx\Big)\Big\}\right).$$

By (53), we have

$$T_q=\max_{\substack{0\leqq b<q\\(b,q)=1}}\left|\sum_{h=0}^{q-1}m(q,h)\,e(hb/q)\right|\leqq\max_{\substack{0\leqq b<q\\(b,q)=1}}\sum_{h=0}^{q-1}m(q,h)\leqq\max_{\substack{0\leqq b<q\\(b,q)=1}}1=1. \tag{73}$$

Furthermore, by (51), we have

$$\left(\frac{F(u)}{u}\right)'=\frac{uF'(u)-F(u)}{u^2}\leqq 0\quad\text{for}\quad 1\leqq u,$$

thus

$$\frac{F(u)}{u}\ \text{ is decreasing for } 1\leqq u. \tag{74}$$

Hence

$$\int_{qQ_j}^{N_j}x^{-3}\big(F(x)\big)^2dx\leqq\int_{Q_j}^{N_j}x^{-1}\left(\frac{F(x)}{x}\right)^2dx\leqq \tag{75}$$

$$\leqq\int_{Q_j}^{N_j}x^{-1}\left(\frac{F(Q_j)}{Q_j}\right)^2dx=\big(F(Q_j)\big)^2Q_j^{-2}\log\frac{N_j}{Q_j}.$$

By (48), (49), (55), (72), (73), (74) and (75), we have

$$\sum_{q=1}^{P}2^{3k+1}k(A+5)^kq(L_q+T_qK)\Big\{L_q^{k-1}N(qQ)^{-2}+ \tag{76}$$

$$+T_q^{k-1}\Big(\big(K+F(N)\big)^{k-1}N^{-1}+K^{k-1}N^k(qQ)^{-(k+1)}+N\int_{qQ}^{N}x^{-3}\big(F(x)\big)^{k-1}dx\Big)\Big\}=$$

$$=O\left(\sum_{q=1}^{P_j}q(M_j+K_j)\Big\{M_j^2N_j(qQ_j)^{-2}+\big(F(N_j)\big)^2N_j^{-1}+K_j^2N_j^3(qQ_j)^{-4}+\right.$$

$$\left.+N_j\big(F(Q_j)\big)^2Q_j^{-2}\log\frac{N_j}{Q_j}\Big\}\right)=O\left((M_j+K_j)\Big\{M_j^2N_jQ_j^{-2}\sum_{q=1}^{P_j}q^{-1}+N_j\left(\frac{F(N_j)}{N_j}\right)^2\sum_{q=1}^{P_j}q+\right.$$

$$\left.+K_j^2N_j^3Q_j^{-4}\sum_{q=1}^{P_j}q^{-3}+P_jN_j\big(F(Q_j)\big)^2Q_j^{-2}\log\frac{N_j}{Q_j}\sum_{q=1}^{P_j}q\Big\}\right)=$$

$$=O\left((M_j+K_j)\Big\{M_j^2N_jQ_j^{-2}\log P_j+N_j\left(\frac{F(Q_j)}{Q_j}\right)^2P_j^2\left(1+\log\frac{N_j}{Q_j}\right)+K_j^2N_j^3Q_j^{-4}\Big\}\right)=$$

$$=O\left((M_j+K_j)\Big(M_j^2N_jQ_j^{-2}\log P_j+N_j\big(F(Q_j)\big)^2Q_j^{-2}P_j^2\log\frac{N_j}{Q_j}+\right.$$

$$\left.+K_j^2N_j^3Q_j^{-4}\Big)\right)=o\big(F^3(N_j)N_j^{-1}\big).$$

Finally, by (51), we have

$$\big(u^{-2}(F(u))^3\big)'=-2u^{-3}\big(F(u)\big)^3+3u^{-2}\big(F(u)\big)^2F'(u)=$$

$$=3u^{-3}\big(F(u)\big)^2\left(uF'(u)-\frac{2}{3}F(u)\right)\geqq 0;$$

thus the function $u^{-2}(F(u))^3$ is increasing for $1\leqq u$. Therefore, with respect to (56) and (74), for $j\to+\infty$ the third term on the right hand side of (14) can be estimated in the following way:

$$\sum_{q=1}^{P} 2(2A+1)^k qT_q^k \int_2^{qQ} x^{-2}(F(x))^3\,dx = O\left(\sum_{q=1}^{P_j} qT_q^3 \int_2^{qQ_j} x^{-2}(F(x))^3\,dx\right) = \tag{77}$$

$$= O\left(\sum_{q=1}^{P_j} qT_q^3 \int_2^{qQ_j} (qQ_j)^{-2}(F(qQ_j))^3\,dx\right) = O\left(\sum_{q=1}^{P_j} qT_q^3\cdot qQ_j\cdot (qQ_j)^{-2}(F(qQ_j))^3\right) =$$

$$= O\left(Q_j^2 \sum_{q=1}^{P_j} q^3T_q^3\cdot\left(\frac{F(qQ_j)}{qQ_j}\right)^3\right) = O\left(Q_j^2 \sum_{q=1}^{P_j} q^3T_q^3\left(\frac{F(Q_j)}{Q_j}\right)^3\right) =$$

$$= O\left((F(Q_j))^3 Q_j^{-1} \sum_{q=1}^{P_j} T_q^3 q^3\right) = o\left((F(Q_j))^3 Q_j^{-1}\frac{Q_j}{N_j}\left(\frac{F(N_j)}{F(Q_j)}\right)^3\right) = o((F(N_j))^3 N_j^{-1}).$$

By (68), (71), (76) and (77), for $j\to+\infty$ the right hand side of (14) is little o of the left hand side, thus for sufficiently large j, (14) holds. Thus in fact, for large j Theorem 2 can be applied and we obtain that (for large j) (62) can be solved.

In order to complete the proof of Theorem 9, we have to show that (iv) can be replaced by (iv′) while (v) can be replaced by (v′) or (v″).

We used (iv) only in the proof of (67); but by (49) and Theorem 6, for $j>j_0$ (67) follows also from (iv′) thus in fact, (iv) can be replaced by (iv′).

By Theorem 7, (63) implies that for $P_j<q\leqq Q_j$, $(b,q)=1$ and $0<u\leqq N_j$ we have

$$|S_u(b/q)| = \left|\sum_{a_i\leqq u} d_i e(a_i b/q)\right| =$$

$$= \left|\sum_{h=0}^{q-1}\Big(\sum_{\substack{a_i\leqq u\\ a_i\equiv h \pmod q}} d_i\Big)e(hb/q)\right| = \left|\sum_{h=0}^{q-1} D(u,q,h)e(hb/q)\right| <$$

$$< 2\delta^{-1}q^{-1}\max_{\substack{0\leqq h<q\\ (h,q)=1}} \sum_{t=0}^{q-1}\left|\sum_{t-\delta q<l\leqq t} D(u,q,hl)-\delta\sum_{j=0}^{q-1} D(u,q,j)\right| =$$

$$= 2\delta^{-1}q^{-1}\max_{\substack{0\leqq h<q\\ (h,q)=1}} \sum_{t=0}^{q-1}\left|\sum_{t-\delta q<l\leqq t} D(u,q,hl)-\delta D(u)\right| \leqq 2\delta^{-1}q^{-1}\frac{\delta q}{2}H_j = H_j.$$

Thus (63) implies (60), which proves that (v) can be replaced by (v′).

In order to complete the proof of Theorem 9, it is sufficient to show that (v″) implies the existence of numbers $H_1, H_2, \dots$ satisfying (v′).

Assume first that

$$q^{-1/2}\log\log q\left(\log q\sum_{\chi\neq\chi_0}\left|\sum_{k=0}^{q-1}\chi(k)D(u,q,k)\right| + D(u)\right) + \sum_{\substack{a_i\leqq u\\ (a_i,q)>1}} d_i < G_j. \tag{78}$$

By using Theorem 8 with $\delta=1/6$ and

$$z_n = \begin{cases} D(u,q,n) & \text{for } (n,q)=1\\ 0 & \text{for } (n,q)>1\end{cases}$$

(note that $q>2$ follows from (47) and $P_j<q$), we obtain for $t=0, \pm1, \pm2, \dots$ and $(h,q)=1$ that

$$\left|\sum_{t-\delta q<l\leqq t} D(u,q,hl)-\delta D(u)\right| = \Bigg|\Bigg(\sum_{\substack{t-\delta q<l\leqq t\\ (l,q)=1}} D(u,q,hl) - \delta\sum_{\substack{a_i\leqq u\\ (a_i,q)=1}} d_i\Bigg) + \tag{79}$$

$$+ \sum_{\substack{t-\delta q<l\leqq t\\ (l,q)>1}}\ \sum_{\substack{a_i\leqq u\\ a_i\equiv hl \pmod q}} d_i - \delta\sum_{\substack{a_i\leqq u\\ (a_i,q)>1}} d_i\Bigg| \leqq \left|\sum_{\substack{t-\delta q<l\leqq t\\ (l,q)=1}} D(u,q,hl)-\delta\sum_{\substack{a_i\leqq u\\ (a_i,q)=1}} d_i\right| +$$

$$+ \sum_{\substack{a_i\leqq u\\ (a_i,q)>1}} d_i + \delta\sum_{\substack{a_i\leqq u\\ (a_i,q)>1}} d_i \leqq c_1 q^{-1/2}\log\log q\left(\log q\sum_{\chi\neq\chi_0}\left|\sum_{k=0}^{q-1}\chi(k)D(u,q,k)\right| + \sum_{\substack{a_i\leqq u\\ (a_i,q)=1}} d_i\right) +$$

$$+(1+\delta)\sum_{\substack{a_i\leqq u\\ (a_i,q)>1}} d_i < c_1 q^{-1/2}\log\log q\left(\log q\sum_{\chi\neq\chi_0}\left|\sum_{k=0}^{q-1}\chi(k)D(u,q,k)\right| + D(u)\right) +$$

$$+2\sum_{\substack{a_i\leqq u\\ (a_i,q)>1}} d_i < (c_1+2)\left\{q^{-1/2}\log\log q\left(\log q\sum_{\chi\neq\chi_0}\left|\sum_{k=0}^{q-1}\chi(k)D(u,q,k)\right| + D(u)\right) +\right.$$

$$\left. + \sum_{\substack{a_i\leqq u\\ (a_i,q)>1}} d_i\right\} < c_6 G_j.$$

Assume now that

$$D(u) < G_j. \tag{80}$$

Then for $t=0, \pm1, \pm2, \dots, (h,q)=1$ we have

$$\left|\sum_{t-\delta q<l\leqq t} D(u,q,hl)-\frac{1}{6}D(u)\right| \leqq \sum_{t-\delta q<l\leqq t}\ \sum_{\substack{a_i\leqq u\\ a_i\equiv hl \pmod q}} d_i + \frac{1}{6}D(u) \leqq \tag{81}$$

$$\leqq \sum_{a_i\leqq u} d_i + \frac{1}{6}D(u) \leqq D(u)+\frac{1}{6}D(u) < 2D(u) < 2G_j.$$

By (79) and (81), both (78) and (80) imply that

$$\left|\sum_{t-\delta q<l\leqq t} D(u,q,hl)-\frac{1}{6}D(u)\right| < c_7 G_j$$

(where $c_7=\max\{c_6, 2\}$) hence

$$\sum_{t=0}^{q-1}\left|\sum_{t-\delta q<l\leqq t} D(u,q,hl)-\frac{1}{6}D(u)\right| < \sum_{t=0}^{q-1} c_7 G_j = \frac{q}{12}\cdot 12c_7 G_j$$

thus (63) in (v′) holds with $\delta=1/6$, $H_j=12c_7G_j$. Furthermore, (65) implies obviously that this number H_j satisfies (61). Thus in fact, (v″) implies (v′) and this completes the proof of Theorem 9.

5. In this section, we show that Theorem 9 involves Goldbach's problem (concerning the representability of large odd numbers as the sum of three primes).

COROLLARY 1. *If N is an odd integer and $N>N_0$ then N can be represented as the sum of three prime numbers.*

Proof. Throughout the proof, we use the following notations:

$$\Lambda(n)=\begin{cases}\log p & \text{if } n=p^\alpha\\ 0 & \text{otherwise,}\end{cases}$$

$$\psi(u,q,a)=\sum_{\substack{n\leqq u\\ n\equiv a \pmod q}}\Lambda(n),\quad \psi(u,\chi)=\sum_{n\leqq u}\chi(n)\Lambda(n)$$

and

$$\psi(u)=\psi(u,1,0)=\sum_{n\leqq u}\Lambda(n)$$

(where q is a positive integer, a an integer, u a real number, χ a character modulo q). Then by the prime number theorem (or a more elementary theorem)

$$\Big|\psi(u,q,a)-\sum_{\substack{p\leqq u\\ p\equiv a \pmod q}}\log p\Big|=\Big|\sum_{\substack{p^\alpha\leqq u,\,\alpha\geqq 2\\ p^\alpha\equiv a \pmod q}}\log p\Big|=O\Big(\sum_{p^2\leqq u}\log p\Big)=O(\sqrt{u}) \tag{82}$$

(uniformly for all a, q), in particular,

$$\Big|\psi(u)-\sum_{p\leqq u}\log p\Big|=O(\sqrt{u}). \tag{83}$$

We need the following lemmas:

Lemma 5. *There exist absolute constants c_8, c_9, x_0 such that for $x>x_0$, we have*

$$|\psi(x)-x|<c_8xe^{-c_9\sqrt{\log x}}.$$

This lemma is equivalent to the prime number theorem; see e.g. [3], pp. 61—70. We need also Siegel's theorem (see e.g. [3], p. 144):

Lemma 6. *If A is an arbitrary large but fixed positive number then there exist absolute constants c_{10}, c_{11}, x_1 such that if $x>x_1$, q, a are any integers satisfying $1\leqq q\leqq$ $\leqq(\log x)^A$ and $(a,q)=1$ then we have*

$$\left|\psi(x,q,a)-\frac{x}{\varphi(k)}\right|<c_{10}xe^{-c_{11}\sqrt{\log x}}.$$

Lemma 7. *If p is a prime number and α is a positive integer then we have*

$$\sum_{\substack{0\leqq a<p^\alpha\\ (a,q)=1}}e(a/p^\alpha)=\mu(p^\alpha)=\begin{cases}-1 & \text{if } \alpha=1\\ 0 & \text{if } \alpha>1.\end{cases} \tag{84}$$

Proof. We have

$$\sum_{\substack{0\leqq a<p^\alpha\\ (a,q)=1}}e(a/p^\alpha)=\sum_{a=0}^{p^\alpha-1}e(a/p^\alpha)-\sum_{b=0}^{p^{\alpha-1}-1}e(bp/p^{\alpha-1})=$$

$$=0-\sum_{b=0}^{p^{\alpha-1}-1}e(b/p^{\alpha-1})=\begin{cases}-1 & \text{if } \alpha=1\\ 0 & \text{if } \alpha>1\end{cases}$$

which proves (84).

Lemma 8. *If q, M are integers such that $1\leqq q\leqq M$ then we have*

$$\sum_\chi|\psi(M,\chi)|<c_{12}(M^{1/2}q+M^{5/7}q^{5/7}+M)(\log M)^{16}.$$

This estimate can be derived easily from the explicit formula for $\psi(M,\chi)$ and zero density estimates for L-functions; see [1], p. 143. (In fact, we avoid the application of Vinogradov's fundamental lemma by using Lemma 8; for a derivation of Vinogradov's lemma from Lemma 8, see [1], Chapter 16.)

Now we are ready to complete the proof of Corollary 1.

We apply Theorem 9 with (iv′) and (v″), respectively, in place of (iv) and (v). We choose the parameters in the theorem in the following way:

We put $a_1=p_1, a_2=p_2, \dots, a_i=p_i, \dots$ (where p_i denotes the i^{th} prime number), $d_i=\log p_i$, $N_j=2(j+j_0)+1$ (where j_0 is a sufficiently large integer), $P_j=[(\log N_j)^{43}]$ and $Q_j=[N_j(\log N_j)^{-44}]$.

Then (49) and for large j_0 and all j, (47) and (48) hold trivially.

Furthermore, we have

$$\delta_j=\min_{p\leqq N_j}\log p=\log 2>\frac{1}{2}, \tag{85}$$

$$\Delta_j=\max_{p\leqq N_j}\log p\leqq\log N \tag{86}$$

and with respect to (82),

$$|D(u,q,a)-\psi(u,q,a)|=\Big|\sum_{\substack{p\leqq u\\ p\equiv a \pmod q}}\log p-\psi(u,q,a)\Big|=O(\sqrt{u}) \tag{87}$$

(uniformly for all a, q), in particular,

$$|D(u)-\psi(u)|=O(\sqrt{u}). \tag{88}$$

If j_0 is sufficiently large, then Lemma 5 and (88) yield for all j and $1\leqq u\leqq N_j$ that

$$|D(u)-u|\leqq|D(u)-\psi(u)|+|\psi(u)-u|< \tag{89}$$

$$<2c_8N_je^{-c_9\sqrt{\log N_j}}<N_je^{-c_{13}\sqrt{\log N_j}}$$

(where we may put $c_{13}=c_9/2$) so that (50) holds with

$$F(u)=u \tag{90}$$

and

$$K_j=N_je^{-c_{13}\sqrt{\log N_j}}; \tag{91}$$

obviously, defining $F(u)$ and K_j in this way, all the conditions in (i) hold.

For $q=1,2,\dots$, $h=0, \pm1, \pm2, \dots$, let

$$m(q,h)=\begin{cases}\dfrac{1}{\varphi(q)} & \text{for } (h,q)=1\\ 0 & \text{for } (h,q)>1.\end{cases}$$

Then (52) and (53) hold trivially. Furthermore, Lemma 6 yields that for all j (provided that j_0 is sufficiently large), $q\leqq P_j$, $(a, q)=1$ and $\sqrt{N_j}<u$ we have

$$\left|\psi(u, q, h)-\frac{u}{\varphi(q)}\right| < c_{14}N_j e^{-c_{15}\sqrt{\log N_j}} \tag{92}$$

and this inequality holds trivially also for $u\leqq\sqrt{N_j}$. By (87), (89) und (90), for $1\leqq q\leqq P_j$ and $0\leqq u\leqq N_j$ we have

$$\sum_{h=0}^{q-1}|D(u, q, h)-m(q, h)D(u)| =$$

$$= \sum_{\substack{0\leqq h<q\\(h,q)=1}}\left|D(u, q, h)-\frac{1}{\varphi(q)}D(u)\right|+\sum_{\substack{0\leqq h<q\\(h,q)>1}}D(u, q, h)\leqq$$

$$\leqq \sum_{\substack{0\leqq h<q\\(h,q)=1}}\left(|D(u, q, h)-\psi(u, q, h)|+\left|\psi(u, q, h)-\frac{u}{\varphi(q)}\right|+\frac{1}{\varphi(q)}|u-D(u)|\right)+$$

$$+\sum_{\substack{p\leqq u\\(p,q)>1}}\log p<\sum_{\substack{0\leqq h<q\\(h,q)=1}}\left(c_{16}\sqrt{N_j}+c_{14}N_je^{-c_{15}\sqrt{\log N_j}}+N_je^{-c_{13}\sqrt{\log N_j}}\right)+$$

$$+\sum_{p|q}\log p<N_je^{-c_{17}\sqrt{\log N_j}}+\log q<N_je^{-c_{18}\sqrt{\log N_j}}.$$

This shows that also the assumption (ii) in Theorem 9 holds with

$$M_j=N_je^{-c_{18}\sqrt{\log N_j}}.$$

Now we are going to show that (55) holds (with the numbers M_j, K_j, P_j, Q_j and function $F(u)$ defined above). In fact, the left hand side of (55) can be estimated in the following way:

$$(M_j+K_j)(M_j^2N_j^2Q_j^{-2}\log P_j+N_j^2(F(Q_j))^2Q_j^{-2}P_j^2\log N_jQ_j^{-1}+K_j^2N_j^4Q_j^{-4})=$$

$$=O\left\{\left(N_je^{-c_{18}\sqrt{\log N_j}}+N_je^{-c_{13}\sqrt{\log N_j}}\right)\left(N_j^2e^{-2c_{18}\sqrt{\log N_j}}N_j^2N_j^{-2}(\log N_j)^{88}\log\log N_j+\right.\right.$$

$$\left.\left.+N_j^2Q_j^2Q_j^{-2}(\log N_j)^{86}\log\log N_j+N_j^2e^{-c_{13}\sqrt{\log N_j}}N_j^4N_j^{-4}(\log N_j)^{176}\right)\right\}=$$

$$=O\left(N_j^3e^{-c_{19}\sqrt{\log N_j}}\right)=o(N_j^3)=o\left((F(N_j))^3\right)$$

which proves (55).

Now we are going to estimate the numbers T_q. By Lemma 7, for $(b, p)=1$ we have

$$T(p^\alpha, b)=\sum_{h=0}^{p^\alpha-1}m(p^\alpha, h)\,e(hb/p^\alpha)= \tag{93}$$

$$=\sum_{\substack{0\leqq h<q\\(h,p)=1}}\frac{1}{\varphi(p^\alpha)}e(hb/p^\alpha)=\frac{1}{\varphi(p^\alpha)}\sum_{\substack{0\leqq h<q\\(h,p)=1}}e(hb/p^\alpha)=\frac{1}{\varphi(p^\alpha)}\sum_{\substack{0\leqq a<q\\(a,p)=1}}e(a/p^\alpha)=\frac{\mu(p^\alpha)}{\varphi(p^\alpha)}$$

hence

$$T_{p^\alpha}=\max_{\substack{0\leqq b<p^\alpha\\(b,p)=1}}|T(p^\alpha, b)|=\max_{\substack{0\leqq b<p^\alpha\\(b,p)=1}}\left|\frac{\mu(p^\alpha)}{\varphi(p^\alpha)}\right|=\frac{|\mu(p^\alpha)|}{\varphi(p^\alpha)}\leqq\frac{1}{\varphi(p^\alpha)}. \tag{94}$$

(94) and Theorem 4 yield with respect to (46) that for $j\to+\infty$

$$\sum_{q=1}^{P_j}T_q^3q^3\leqq\sum_{q=1}^{P_j}\prod_{p^\alpha\|q}T_{p^\alpha}^3p^{3\alpha}=\sum_{q=1}^{P_j}\prod_{p^\alpha\|q}\left(\frac{|\mu(p^\alpha)|}{\varphi(p^\alpha)}\right)^3p^{3\alpha}=\sum_{q=1}^{P_j}\frac{|\mu(q)|q^3}{(\varphi(q))^3}< \tag{95}$$

$$<c_{20}\sum_{q=1}^{P_j}(\log\log(q+2))^3<c_{20}\sum_{q=1}^{P_j}(\log\log(P_j+2))^3=$$

$$=c_{20}P_j(\log\log(P_j+2))^3=O\left((\log N_j)^{43}\log\log\log N_j\right)=o\left((\log N_j)^{44}\right).$$

On the other hand, we have

$$\frac{Q_j}{N_j}\left(\frac{F(N_j)}{F(Q_j)}\right)^3=\frac{Q_j}{N_j}\frac{N_j^3}{Q_j^3}=\frac{N_j^2}{Q_j^2}\geqq\frac{N_j^2}{(N_j(\log N_j)^{-44})^2}=(\log N_j)^{88}. \tag{96}$$

(95) and (96) yield (56) and this proves that also the assumption (iii) in Theorem 9 holds.

Now we are going to show that also (iv′) holds. By (46) and (94), we have

$$\sum_{q=1}^{+\infty}\prod_{p^\alpha\|q}\varphi(p^\alpha)T_{p^\alpha}^3=\sum_{q=1}^{+\infty}\prod_{p^\alpha\|q}\varphi(p^\alpha)\frac{|\mu(p^\alpha)|}{(\varphi(p^\alpha))^3}=$$

$$=\sum_{q=1}^{+\infty}\frac{|\mu(q)|}{(\varphi(q))^2}<\sum_{q=1}^{+\infty}c_{21}\frac{(\log\log(q+2))^2}{q^2}<+\infty$$

and

$$\sum_{\alpha=1}^{+\infty}\varphi(p^\alpha)T_{p^\alpha}^3=\sum_{\alpha=1}^{+\infty}\varphi(p^\alpha)\frac{|\mu(p^\alpha)|}{(\varphi(p^\alpha))^3}=\sum_{\alpha=1}^{+\infty}\frac{|\mu(p^\alpha)|}{(\varphi(p^\alpha))^2}=$$

$$=\frac{1}{(\varphi(p))^2}=\frac{1}{(p-1)^2}\leqq\frac{1}{2^2}=\frac{1}{4}\quad\text{for}\quad p\geqq3$$

thus (57) holds and $\omega=3$ satisfies (58). Furthermore, the only prime number satisfying $p<\omega=3$ is $p=2$, and for $\alpha\geqq\alpha_2=2$, we have

$$B(2^\alpha, N_j)=\sum_{\substack{0\leqq b<2^\alpha\\(b,2)=1}}\left(T(2^\alpha, b)\right)^3e(-N_jb/2^\alpha)= \tag{97}$$

$$=\sum_{\substack{0\leqq b<2^\alpha\\(b,2)=1}}\left(\frac{\mu(2^\alpha)}{\varphi(2^\alpha)}\right)^3e(-N_jb/2^\alpha)=\sum_{\substack{0\leqq b<2^\alpha\\(b,2)=1}}0\cdot e(-N_jb/2^\alpha)=0.$$

Furthermore, for all j, N_j is odd thus by (93) and (97), we have

$$A(2, N_j) = 1+\sum_{\alpha=1}^{+\infty} B(2^\alpha, N_j) = 1+B(2, N_j) =$$

$$= 1+\sum_{\substack{0 \leqq b<2 \\ (b,2)=1}} (T(2, b))^3 e(-N_j b/2) = 1+(T(2, 1))^3 e(-N_j/2) =$$

$$= 1+\left(\frac{\mu(2)}{\varphi(2)}\right)^3 (-1) = 1+1 = 2 \neq 0$$

which proves that also (iv′) holds.

Finally, in order to prove that also (v″) holds, we have to estimate the left hand side of (64).

Assume first that $Q_j<u\leqq N_j$. Then by (83) and Lemma 8, for $P_j<q\leqq Q_j$ we have

$$\sum_{\chi\neq\chi_0}\left|\sum_{k=0}^{q-1}\chi(k)D(u, q, k)\right| = \sum_{\chi\neq\chi_0}\left|\sum_{k=0}^{q-1}\chi(k)\sum_{\substack{p\leqq u \\ p\equiv k \,(\mathrm{mod}\, q)}}\log p\right| =$$

$$= \sum_{\chi\neq\chi_0}\left|\sum_{n\leqq u}\chi(n)\Lambda(n)-\sum_{\substack{p^\alpha\leqq u \\ \alpha\geqq 2}}\chi(p^\alpha)\log p\right| \leqq \sum_{\chi}\left|\sum_{n\leqq u}\chi(n)\Lambda(n)\right|+\sum_{\chi}\sum_{\substack{p^\alpha\leqq u \\ \alpha\geqq 2}}\log p =$$

$$= \sum_{\chi}|\psi(u, \chi)|+\sum_{\chi}\left|\psi(u)-\sum_{p\leqq u}\log p\right| < c_{12}(u^{1/2}q+u^{5/7}q^{5/7}+u)(\log u)^{16}+\sum_{\chi}c_{22}u^{1/2} <$$

$$< c_{12}(u^{1/2}q+u^{5/7}q^{5/7}+u)(\log u)^{16}+c_{22}u^{1/2}q <$$

$$< c_{23}(u^{1/2}q+u^{5/7}q^{5/7}+u)(\log u)^{16} \leqq c_{23}(N_j^{1/2}q+N_j^{5/7}q^{5/7}+N_j)(\log N_j)^{16}$$

hence (with respect to (89))

$$(98) \quad q^{-1/2}\log\log q\left\{\log q\sum_{\chi\neq\chi_0}\left|\sum_{k=0}^{q-1}\chi(k)D(u, q, k)\right|+D(u)\right\}+\sum_{\substack{p\leqq u \\ (p,q)>1}}\log p <$$

$$< c_{24}q^{-1/2}\log\log q\{\log q(N_j^{1/2}q+N_j^{5/7}q^{5/7}+N_j)(\log N_j)^{16}+u\}+\sum_{p|q}\log p \leqq$$

$$\leqq c_{24}\log\log N_j\{\log N_j(N_j^{1/2}q^{1/2}+N_j^{5/7}q^{3/14}+N_jq^{-1/2})(\log N_j)^{16}+N_jq^{-1/2}\}+\log q \leqq$$

$$\leqq c_{25}\log\log N_j(N_j^{1/2}Q_j^{1/2}+N_j^{5/7}Q_j^{3/14}+N_jP_j^{-1/2})(\log N_j)^{17}+\log N_j <$$

$$< c_{26}\log\log N_j\{N_j^{1/2}N_j^{1/2}(\log N_j)^{-22}+N_j^{5/7}N_j^{3/14}(\log N_j)^{-66/7}+N_j(\log N_j)^{-43/2}\}\times$$

$$\times(\log N_j)^{17} < c_{27}\log\log N_j\cdot N_j(\log N_j)^{-43/2}(\log N_j)^{17} =$$

$$= c_{27}N_j(\log N_j)^{-9/2}\log\log N_j < N_j(\log N_j)^{-4} \quad (\text{for } Q_j<u\leqq N_j)$$

while with respect to (89), for $u\leqq Q_j$ we have

$$(99) \quad D(u)\leqq D(Q_j)<2Q_j<2N_j(\log N_j)^{-44}<N_j(\log N_j)^{-4} \quad (\text{for } u\leqq Q_j).$$

(98) and (99) yield (64) with

$$(100) \qquad G_j = N_j(\log N_j)^{-4}.$$

Furthermore, we have

$$(101) \qquad \frac{P_jQ_j}{N_j^2\Delta_j^2}F^2(N_j) \geqq \frac{(\log N_j)^{43}N_j(\log N_j)^{-44}}{N_j^2(\log N_j)^2}N_j^2 = N_j(\log N_j)^{-3}.$$

(100) and (101) obviously imply that also (65) holds and this completes the proof of Corollary 1.

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ON WALSH SERIES WITH COEFFICIENTS TENDING MONOTONICALLY TO ZERO

By

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1. We consider the Walsh orthonormal system $\{w_n(x)\}_{n=0}^{\infty}$ defined on $[0, 1]$ in the Paley enumeration. In this case the Walsh functions are defined with the aid of of the Rademacher functions

$$r_\nu(x) = \operatorname{sign} \sin 2^\nu \pi x \quad (\nu = 0, 1, 2, \ldots;\ 0 \leqq x \leqq 1)$$

in the following way (see, e.g. [1, p. 60]): if $n=0$ set $w_0(x)\equiv 1$, and if $n\geqq 1$ and

$$n = \sum_{i=1}^{p} 2^{\nu_i}, \quad 0 \leqq \nu_1 < \nu_2 < \ldots < \nu_p,$$

is the dyadic representation of n, then set

$$w_n(x) = \prod_{i=1}^{p} r_{\nu_i+1}(x).$$

We shall study the integrability properties of the maximum of the partial sums of the series

$$\sum_{n=0}^{\infty} a_n w_n(x), \tag{1}$$

where the sequence $\{a_n\}$ of coefficients monotonically decreases to zero. In this paper the integral is always meant in the sense of Lebesgue.

It is well-known that in the case $a_0 \geqq a_1 \geqq a_2 \geqq \ldots \to 0$ series (1) converges to a function $f(x)$ (except perhaps at $x=0$), and the convergence is uniform on every interval $[\delta, 1]$, $\delta>0$ (cf. Šneider [12]).

The following three statements, due to Balašov [3], Yano [14] and Fomin [6] (in this order), give conditions ensuring that the sum $f(x)$ of series (1) should belong to $L(0, 1)$.

Theorem A. *Assume that* $a_0 \geqq a_1 \geqq a_2 \geqq \ldots \to 0$.

(i) *If*

$$\sum_{n=1}^{\infty} \frac{a_n}{n} < \infty,$$

then the sum $f(x)$ of series (1) *is integrable and* (1) *is the Walsh—Fourier series of $f(x)$.*

(ii) *If the sequence* $\{a_n\}$ *is quasi-convex, i.e. if*

$$\sum_{n=0}^{\infty}(n+1)|\Delta^2 a_n| < \infty, \tag{2}$$

where

$$\Delta^2 a_n = \Delta a_n - \Delta a_{n+1} \quad \text{and} \quad \Delta a_n = a_n - a_{n+1} \quad (n = 0, 1, 2, \ldots),$$

then the conclusion of (i) *holds true.*

(iii) *Under conditions of* (ii) *we have*

$$\int_0^1 \left| f(x) - \sum_{n=0}^{m} a_n w_n(x) \right| dx \to 0 \quad \text{as} \quad m \to \infty$$

if and only if $a_n = o(1/\log n)$.

In this paper all logarithms are of base 2.

We note that for cosine and sine series the corresponding statements are well-known (see, e.g. [15, Vol. I, pp. 183—185]).

2. The following result, proved by HARDY and LITTLEWOOD [9] (see also [15, Vol. II, p. 129]), characterizes the case when the sums

$$g(x) = \sum_{n=1}^{\infty} a_n \cos nx \quad \text{and} \quad h(x) = \sum_{n=1}^{\infty} a_n \sin nx \tag{3}$$

$(a_1 \geqq a_2 \geqq \ldots \to 0)$ belong to $L^r(-\pi, \pi)$ for a given $r>1$.

THEOREM B. (i) *If* $a_1 \geqq a_2 \geqq \ldots \geqq 0$, *then a necessary and sufficient condition that the sum* $g(x)$ *of series* (3i) *belong to* $L^r(-\pi, \pi)$, $r>1$, *is that*

$$\sum_{n=1}^{\infty} a_n^r n^{r-2} < \infty. \tag{4}$$

If this condition is satisfied, then

$$\int_{-\pi}^{\pi} |g(x)|^r dx \leqq C \sum_{n=1}^{\infty} a_n^r n^{r-2},$$

where C is a constant, depending at most on r.

(ii) *The result holds also for series* (3ii).

In the sequel C means a positive constant, depending at most on r, but not necessarily the same at each occurrence.

Our main goal is to extend this result for Walsh series in a more general setting. The sufficiency part is contained in the following

THEOREM 1. *If* $a_0 \geqq a_1 \geqq a_2 \geqq \ldots \geqq 0$ *and condition* (4) *is satisfied for a given* $r \geqq 1$, *then*

$$\int_0^1 \left(\sup_{m \geqq k} \left| \sum_{n=k}^{m} a_n w_n(x) \right| \right)^r dx \to 0 \quad \text{as} \quad k \to \infty, \tag{5}$$

and, in particular,

$$\int_0^1 \left(\sup_{m \geqq 0} \left| \sum_{n=0}^{m} a_n w_n(x) \right| \right)^r dx \leqq C \left(a_0^r + \sum_{n=1}^{\infty} a_n^r n^{r-2} \right). \tag{6}$$

In the proof of Theorem 1 we need an inequality due to HARDY and LITTLEWOOD [8], expressed in the following

LEMMA 1. *If* $b_n \geqq 0$ $(n=1, 2, \ldots)$, $c>1$ *and* $r \geqq 1$, *then*

$$\sum_{\nu=1}^{\infty} \frac{1}{\nu^c} \left(\sum_{n=1}^{\nu} b_n \right)^r \leqq r^r \sum_{n=1}^{\infty} b_n^r n^{r-c}.$$

PROOF OF THEOREM 1. We shall show that for $k \geqq 1$

$$I_k = \int_0^1 \left(\sup_{m \geqq k} \left| \sum_{n=k}^{m} a_n w_n(x) \right| \right)^r dx \leqq \tag{7}$$

$$\leqq \begin{cases} C\left(a_k \max(1, \log k) + \sum\limits_{n=2k}^{\infty} a_n n^{-1} \right) & \text{if} \quad r = 1, \\ C\left(a_k^r k^{r-1} + \sum\limits_{n=2k}^{\infty} a_n^r n^{r-2} \right) & \text{if} \quad 1 < r < 2, \\ C \sum\limits_{n=k}^{\infty} a_n^r n^{r-2} & \text{if} \quad r \geqq 2. \end{cases}$$

It is not hard to verify that (7) implies both (5) and (6). Indeed, e.g. in case $r=1$

$$a_k \log k \leqq C \sum_{n=[\sqrt{k}\,]}^{k} \frac{a_n}{n} \to 0 \quad \text{as} \quad k \to \infty.$$

To begin with, denote

$$D_m(x) = \sum_{n=0}^{m} w_n(x) \quad (m = 0, 1, 2, \ldots)$$

the Dirichlet kernel. Applying the usual Abel transformation and the known inequality

$$|D_m(x)| \leqq \frac{2}{x} \quad (m = 0, 1, 2, \ldots; \; 0 < x \leqq 1)$$

(see, e. g. FINE [5, p. 391]) we get that

$$\left| \sum_{n=l}^{m} a_n w_n(x) \right| \leqq \left| \sum_{n=l}^{m-1} D_n(x) \Delta a_n \right| + |a_l D_{l-1}(x)| + \tag{8}$$

$$+ |a_m D_m(x)| \leqq \frac{4a_l}{x} \quad (0 \leqq l \leqq m; \; 0 < x \leqq 1).$$

Now suppose that $1/(\nu+1) \leqq x \leqq 1/\nu$, $\nu \geqq 1$. Then, by the monotonicity of $\{a_n\}$ and (8), we have

$$\sup_{m\geqq k}\left|\sum_{n=k}^{m} a_n w_n(x)\right| \leqq \sum_{n=k}^{k+\nu-1} a_n + \sup_{m\geqq k+\nu}\left|\sum_{n=k+\nu}^{m} a_n w_n(x)\right| \leqq$$

$$\leqq \sum_{n=k}^{k+\nu-1} a_n + \frac{4a_{k+\nu}}{x} \leqq \sum_{n=k}^{k+\nu-1} a_n + 8\nu a_{k+\nu} \leqq 9\sum_{n=k}^{k+\nu-1} a_n,$$

whence it follows that

$$I_k = \sum_{\nu=1}^{\infty}\int_{1/(\nu+1)}^{1/\nu}\left(\sup_{m\geqq k}\left|\sum_{n=k}^{m} a_n w_n(x)\right|\right)^r dx \leqq 9\sum_{\nu=1}^{\infty}\frac{1}{\nu^2}\left(\sum_{n=k}^{k+\nu-1} a_n\right)^r.$$

By virtue of Lemma 1,

$$I_k \leqq 9r^r \sum_{n=k}^{\infty} a_n^r (n-k+1)^{r-2}.$$

This obviously implies (7) in the case $r \geqq 2$, while in the case $1 \leqq r < 2$ we can finish the proof as follows:

$$\left(\sum_{n=k}^{2k-1} + \sum_{n=2k}^{\infty}\right) a_n^r (n-k+1)^{r-2} \leqq a_k^r \sum_{l=1}^{k} l^{r-2} + 2^{2-r}\sum_{n=2k}^{\infty} a_n^r n^{r-2}.$$

The proof of Theorem 1 is complete.

3. Now we turn to the necessity part. We distinguish two cases according as $r > 1$ or $r = 1$.

THEOREM 2. *If $a_0 \geqq a_1 \geqq a_2 \geqq \ldots \geqq 0$ and the sum $f(x)$ of series* (1) *belongs to $L^r(0, 1)$ for a given $r > 1$, then* (4) *is satisfied.*

From Theorems 1 and 2 we can conclude the following two corollaries.

COROLLARY 1. *Statement* (i) *in Theorem* B *holds also for Walsh series* (1).

COROLLARY 2. *If $a_0 \geqq a_1 \geqq a_2 \geqq \ldots \geqq 0$ and $r > 1$, then the maximum of the partial sums of series* (1) *belongs to $L^r(0, 1)$ if and only if* (1) *is the Walsh—Fourier series of a function belonging to $L^r(0, 1)$.*

The statement of Corollary 2 without any restrictions on the coefficients $\{a_n\}$ was proved by SJÖLIN [11] using much deeper tools.

The fact that if the sum $f(x)$ of series (1) (in the case $a_0 \geqq a_1 \geqq a_2 \geqq \ldots \geqq 0$) belongs to $L^r(0, 1)$ for some $r \geqq 1$, then (1) is the Walsh—Fourier series of $f(x)$, is a particular case of the following general result of ARUTJUNJAN and TALALJAN [2].

LEMMA 2. *If $a_n \to 0$ as $n \to \infty$ and there exists a sequence $n_1 < n_2 < \ldots$ of positive integers such that the subsequence $\left\{\sum_{n=0}^{2^{n_i}} a_n w_n(x)\right\}$ of the partial sums of series* (1) *converges to a finite function $f(x) \in L(0, 1)$ everywhere, except possibly on a denumerable set, then* (1) *is the Walsh—Fourier series of $f(x)$.*

PROOF OF THEOREM 2. The proof is similar to that of the necessity part of Theorem B. Set

$$F(u) = \int_0^u f(x)\,dx \quad \text{and} \quad F_1(u) = \int_0^u |f(x)|\,dx.$$

It is clear that for $\mu \geqq 1$

$$D_{2^\mu-1}(x) = \prod_{\nu=1}^{\mu}(1+r_\nu(x)) = \begin{cases} 2^\mu & \text{for} \quad x \in \left(0, \frac{1}{2^\mu}\right), \\ 0 & \text{for} \quad x \in \left(\frac{1}{2^\mu}, 1\right). \end{cases} \tag{9}$$

Thus, by Lemma 2 and the monotonicity of $\{a_n\}$,

$$F\left(\frac{1}{2^\mu}\right) = \frac{1}{2^\mu}\int_0^1 f(x) D_{2^\mu-1}(x)\,dx = \frac{1}{2^\mu}\sum_{n=0}^{2^\mu-1} a_n \geqq a_{2^\mu-1} \quad (\mu = 1, 2, \ldots).$$

An easy calculation gives that

$$\sum_{\mu=1}^{\infty} a_{2^\mu-1}^r 2^{\mu(r-1)} \leqq \sum_{\mu=1}^{\infty} F^r\left(\frac{1}{2^\mu}\right) 2^{\mu(r-1)} \leqq \tag{10}$$

$$\leqq 2^r \sum_{\mu=1}^{\infty}\int_{1/2^\mu}^{1/2^{\mu-1}}\left\{\frac{F_1(x)}{x}\right\}^r dx = 2^r\int_0^1\left\{\frac{F_1(x)}{x}\right\}^r dx.$$

Now we make use of an inequality of HARDY [7] (cf. [15, Vol. I, p. 20]) which provides that

$$\int_0^1\left\{\frac{F_1(x)}{x}\right\}^r dx \leqq \left(\frac{r}{r-1}\right)^r\int_0^1 |f(x)|^r dx. \tag{11}$$

Here the condition $r > 1$ is essential.

Taking into account (10) and (11) we arrive at the wanted inequality (4):

$$\sum_{n=1}^{\infty} a_n^r n^{r-2} \leqq \sum_{\mu=1}^{\infty} a_{2^\mu-1}^r \sum_{n=2^\mu-1}^{2^{\mu+1}-2} n^{r-2} \leqq$$

$$\leqq \max(2^{r-2}, 2^{2-r})\sum_{\mu=1}^{\infty} a_{2^\mu-1}^r 2^{\mu(r-1)} \leqq C\int_0^1 |f(x)|^r dx.$$

This completes the proof of Theorem 2.

4. In the case $r = 1$ the necessity part is covered by the following results of LADHAWALA [10] and BURKHOLDER [4].

THEOREM C [10]. *There exists a constant C such that for every sequence* $\{a_n\}$ *of coefficients we have*

$$\sum_{n=1}^{\infty} \frac{|a_n|}{n} \leqq C \int_0^1 \left(\sum_{k=1}^{\infty} \left[\sum_{n=2^{k-1}}^{2^k-1} a_n w_n(x) \right]^2 \right)^{1/2} dx.$$

THEOREM D [4, on p. 33 for the special case $\Phi(x)=x$]. *There exist constants* C_1 *and* C_2 *such that for every martingale* $\{f_k\}_{k=1}^{\infty}$ *we have*

$$C_1 E\left(\sum_{k=1}^{\infty} [f_k - f_{k-1}]^2 \right)^{1/2} \leqq E\left(\sup_{k \geqq 1} |f_k|\right) \leqq C_2 E\left(\sum_{k=1}^{\infty} [f_k - f_{k-1}]^2 \right)^{1/2} \quad (f_0 \equiv 0).$$

We only have to take into account that $\left\{f_k = \sum_{n=0}^{2^k-1} a_n w_n(x)\right\}$ is a martingale in order to conclude the following inequality:

$$\sum_{n=1}^{\infty} \frac{|a_n|}{n} \leqq C \int_0^1 \left(\sup_{k \geqq 1} \left| \sum_{n=0}^{2^k-1} a_n w_n(x) \right| \right) dx. \tag{12}$$

Comparing Theorem 1 for the case $r=1$ and (12) we can state the following

COROLLARY 3. *If* $a_0 \geqq a_1 \geqq a_2 \geqq \ldots \geqq 0$, *the maximum of the partial sums of series* (1) *is integrable if and only if* $\sum a_n/n < \infty$.

Now consider the special choice $a_n = \{\log(n+4) \log\log(n+4)\}^{-1}$. The combination of Theorem A (iii) and (12) provides

COROLLARY 4. *There exists an integrable function* $f(x)$, *the partial sums of the Walsh—Fourier series of which converge to* $f(x)$ *in* $L(0, 1)$-*metric, but the maximum of the partial sums is not integrable.*

REMARK. The corresponding results for cosine and sine series with coefficients tending monotonically to zero were established by STEČKIN [13].

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ON MOSAICS CONSISTING OF MULTIDIMENSIONAL CROSSES

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I. Introduction

Let us reflect an n-dimensional cube in all its faces. The $2n+1$ cubes obtained form a so called n-dimensional cross. F. KÁRTESZI [1] raised the question whether it is possible to construct a mosaic from n-dimensional crosses filling up the n-dimensional space. The question is partially answered by the following theorem of E. MOLNÁR [2].

THEOREM A. *Let us consider lattice-like mosaics of n-dimensional crosses where the neighbouring ones join each other along entire $(n-1)$-dimensional cube faces. The number of pairwise non-congruent mosaics of this type and the number of pairwise non-isomorphic Abelian groups of order $2n+1$, are equal.*

The question whether mosaics consisting of n-dimensional crosses not always join each other along entire cube faces, and, whether non-lattice-like mosaics of crosses exist, are still unsolved.

The surprising fact that for certain numbers n, a lattice-like mosaic of n-dimensional crosses join each other not along entire cube faces, has been proved in [3].

Now we state the following two theorems.

THEOREM 1. *In all lattice-like mosaics of n-dimensional crosses, adjacent crosses always join each other along entire cube faces if $2n+1$ is a prime number. If $2n+1$ $(n\geqq 1)$ is not prime, then there exists a lattice-like cross mosaic in which n-dimensional crosses join each other along no entire cube faces.*

THEOREM 2. *If $2n+1$ $(n\geqq 1)$ is not a prime number then a non-lattice-like mosaic of n-dimensional crosses exists.*

In proving Theorem 1, first we show that it suffices to consider only lattices with rational coordinates. In case of rational lattices, the problem may be reformulated into group theoretical terminology, rather facilitated by the work of G. HAJÓS [4].

The group theoretical proof of Theorem 1 will be based on a famous theorem by L. RÉDEI [5] on the group factorization. The second surprising result relies on a special factorization of cyclic Abelian groups.

II. Notations and preliminaries

Let $\mathcal{E}^n$ denote the n-dimensional Euclidean space. Translations of $\mathcal{E}^n$ belong to the n-dimensional vector space $\mathbf{E}^n$ over the real number field R. Let $\mathbf{e}_1, \dots, \mathbf{e}_n$ be a fixed orthonormal basis in $\mathbf{E}^n$. Take the set

$$\mathbf{C} := \{\mathbf{c} : \mathbf{c} = c_1\mathbf{e}_1 + \dots + c_n\mathbf{e}_n;\ c_1, \dots, c_n \in R;\ 0 \leqq c_1 \leqq 1, \dots, 0 \leqq c_n \leqq 1\} \subset \mathbf{E}^n.$$

Let $0 \in \mathscr{E}^n$ be a fixed point. The set

$$\mathscr{C}_0 := \{C : \overrightarrow{OC} \in \mathbf{C}\} \subset \mathscr{E}^n$$

will be called cube, with preferential vertex 0. From the definition it is clear that $\mathscr{C}_0$ is a convex closed set (in the usual topology of $\mathscr{E}^n$). Let int $\mathscr{C}_0$ denote the set of inner points of the cube $\mathscr{C}_0$, i.e.

$$\operatorname{int} \mathscr{C}_0 := \{C : \overrightarrow{OC} = c_1\mathbf{e}_1 + \ldots + c_n\mathbf{e}_n;$$

$$c_1, \ldots, c_n \in R;\ 0 < c_1 < 1, \ldots, 0 < c_n < 1\} \subset \mathscr{E}^n.$$

The following statement is readily seen. Let $P, Q \in \mathscr{E}^n$ and $\overrightarrow{PQ} = p_1\mathbf{e}_1 + \ldots + p_n\mathbf{e}_n$. Then $\operatorname{int} \mathscr{C}_P \cap \operatorname{int} \mathscr{C}_Q \neq \emptyset$ if and only if $|p_1| < 1, \ldots, |p_n| < 1$.

For $\mathbf{K}^n := \{0, \mathbf{e}_1, \ldots, \mathbf{e}_n, -\mathbf{e}_1, \ldots, -\mathbf{e}_n\} \subset \mathbf{E}^n$, the n-dimensional cross of the preferential vertex 0 is interpreted by

$$\bar{\mathscr{K}}_0^n := \bigcup_{\overrightarrow{OP} \in \mathbf{K}^n} \mathscr{C}_P \subset \mathscr{E}^n.$$

A cross is sometimes considered as a set consisting of cubes. Then also, it is called a cross, but the difference is indicated by the notation

$$\mathscr{K}_0^n := \{\mathscr{C}_P : \overrightarrow{OP} \in \mathbf{K}^n\}.$$

A set Ω of cubes is called a cube mosaic in the space $\mathscr{E}^n$ if

1. $\bigcup_{\mathscr{C} \in \Omega} \mathscr{C} \supseteq \mathscr{E}^n$,
2. $\operatorname{int} \mathscr{C} \cap \operatorname{int} \mathscr{C}' = \emptyset$ if $\mathscr{C} \neq \mathscr{C}'$ and $\mathscr{C}, \mathscr{C}' \in \Omega$.

A set Γ of crosses is called a cross mosaic, if

$$\Omega := \{\mathscr{C} : \mathscr{C} \in \mathscr{K}_P^n,\ \mathscr{K}_P^n \in \Gamma\}$$

is a cube mosaic. By convention, if Ω is a set of cubes, then we denote

$$\bar{\Omega} := \bigcup_{\mathscr{C} \in \Omega} \mathscr{C} \subseteq \mathscr{E}^n.$$

Let a set of vectors $\mathbf{L} \subseteq \mathbf{E}^n$ be an Abelian group with the operation of vector addition. Consider the set of points

$$(0, \mathbf{L}) := \{P \in \mathscr{E}^n : \overrightarrow{OP} \in \mathbf{L}\} \subseteq \mathscr{E}^n$$

with the origin 0.

The vector set $\mathbf{L}$ is called an m-dimensional lattice if

1. $(0, \mathbf{L})$ is a discrete set (in the usual topology of $\mathscr{E}^n$),
2. There exists an m-dimensional subspace $\mathbf{A} \subseteq \mathbf{E}^n$ such that $\mathbf{L} \subseteq \mathbf{A}$, but for all $(m-1)$-dimensional subspace $\mathbf{A}' \subseteq \mathbf{E}^n$, $\mathbf{L} \nsubseteq \mathbf{A}'$ holds.

By definition, if $\mathbf{L} \subseteq \mathbf{E}^n$ is a lattice, then all subgroups of $\mathbf{L}$ are lattices. The following theorems will be used :

THEOREM 3. *If $\mathbf{L} \subseteq \mathbf{E}^n$ is an m-dimensional lattice, then there exist vectors $\mathbf{l}_1, \ldots, \mathbf{l}_m \in \mathbf{E}^n$ such that*

$$\mathbf{L} = \{\mathbf{l} : \mathbf{l} = x_1\mathbf{l}_1 + \ldots + x_m\mathbf{l}_m;\ x_1, \ldots, x_m \in Z\}.$$

(Z being the set of integer numbers.)

We say that the vectors $\mathbf{l}_1, \ldots, \mathbf{l}_m$ span the lattice $\mathbf{L}$.

THEOREM 4. *Let $\mathbf{L} \subseteq \mathbf{E}^n$ be an m-dimensional lattice, and $\mathbf{A} \subseteq \mathbf{E}^n$ a subspace. If $\mathbf{L} \cap \mathbf{A}$ is an s-dimensional lattice spanned by the vectors $\mathbf{l}_1, \ldots, \mathbf{l}_s$ then there exist $\mathbf{l}_{s+1}, \ldots, \mathbf{l}_m$ such that $\mathbf{l}_1, \ldots, \mathbf{l}_s, \mathbf{l}_{s+1}, \ldots, \mathbf{l}_m$ span the lattice $\mathbf{L}$.*

Proof of Theorems 3 and 4 are found in [4].

The lattice

$$\mathbf{N} := \{\mathbf{l} : \mathbf{l} = x_1\mathbf{e}_1 + \ldots + x_n\mathbf{e}_n;\ x_1, \ldots, x_n \in Z\}$$

with the fixed orthonormal basis $\mathbf{e}_1, \ldots, \mathbf{e}_n$ is called basic lattice. The sets

$$(0, \mathbf{L}) := \{P : \overrightarrow{OP} \in \mathbf{L}\},\ (\mathscr{C}_0, \mathbf{L}) := \{\mathscr{C}_P : \overrightarrow{OP} \in \mathbf{L}\},\ (\mathscr{K}_0^n, \mathbf{L}) := \{\mathscr{K}_P^n : \overrightarrow{OP} \in \mathbf{L}\}$$

are called point lattice, cube lattice and cross lattice, respectively. A cross lattice $(\mathscr{K}_0^n, \mathbf{L})$ is embedded in the basic lattice $\mathbf{N}$ if $\mathbf{L} \subseteq \mathbf{N}$. Obviously, this means that adjacent crosses can only join along entire cube sides.

If $\mathbf{L}, \mathbf{H} \subseteq \mathbf{E}^n$ are lattices and $\mathbf{H} \subseteq \mathbf{L}$, then the quotient group $\mathbf{L}/\mathbf{H}$ may be formed.

Let $\langle \mathbf{l} \rangle$ be a coset of the quotient group $\mathbf{L}/\mathbf{H}$ represented by the element $\mathbf{l} \in \mathbf{H}$. The following facts are assumed to be known. $\langle \mathbf{l} \rangle = \langle \mathbf{h} \rangle$ if and only if $(\mathbf{l} - \mathbf{h}) \in \mathbf{H}$, $\mathbf{l}, \mathbf{h} \in \mathbf{L}$. By definition $\langle \mathbf{l} \rangle + \langle \mathbf{h} \rangle = \langle \mathbf{l} + \mathbf{h} \rangle$; in particular $u\langle \mathbf{l} \rangle = \langle u\mathbf{l} \rangle$, $u \in Z$. The neutral element of $\mathbf{L}/\mathbf{H}$ is coset $\langle 0 \rangle = \mathbf{H}$. The inverse of $\langle \mathbf{l} \rangle$ is $-\langle \mathbf{l} \rangle = \langle -\mathbf{l} \rangle$.

III. Cube mosaics

The vectors $\mathbf{p}, \mathbf{q} \in \mathbf{E}^n$ are said to be related by α, denoted by $\mathbf{p}\alpha\mathbf{q}$, if $\mathbf{q} - \mathbf{p} = a_1\mathbf{e}_1 + \ldots + a_n\mathbf{e}_n$ and a_1 is a rational number. By means of relation α let us interpret relations β, γ, δ on the pointset $\mathscr{E}^n$, cube set Ω and cross set Γ, respectively. For

$$P, Q \in \mathscr{E}^n,\ \mathscr{C}_P, \mathscr{C}_Q \in \Omega,\ \mathscr{K}_P^n, \mathscr{K}_Q^n \in \Gamma$$

the relations

$$P\beta Q,\ \mathscr{C}_P \gamma \mathscr{C}_Q,\ \mathscr{K}_P^n \delta \mathscr{K}_Q^n$$

hold, respectively if and only if $\overrightarrow{OP}\alpha\overrightarrow{OQ}$.

Obviously, $\alpha, \beta, \gamma, \delta$ are equivalence relations, producing a class decomposition in $\mathbf{E}^n, \mathscr{E}^n, \Omega, \Gamma$, respectively.

Briefly, the equivalence classes of α are called α classes. The same terminology will be used for relations β, γ, δ.

THEOREM 5. *If Ω is a mosaic of cubes in the space $\mathscr{E}^n$ and $\Delta \subseteq \Omega$ is a γ class, then Δ is an infinite prism in the direction $\mathbf{e}_1$. In other terms, if $Q \in \bar{\Delta}$, $\lambda \in R$ and $\overrightarrow{OP} = \overrightarrow{OQ} + \lambda\mathbf{e}_1$ then $P \in \bar{\Delta}$.*

The proof by induction is based on the remark that hyperplane sections of an n-dimensional cube parallel with its sides are $(n-1)$-dimensional cubes. Taking a subspace

$$\mathbf{A}_j := \{\mathbf{a} : \mathbf{a} = a_1\mathbf{e}_1 + \ldots + a_{j-1}\mathbf{e}_{j-1} + a_{j+1}\mathbf{e}_{j+1} + \ldots + a_n\mathbf{e}_n;$$

$$a_1, \ldots, a_{j-1}, a_{j+1}, \ldots, a_n \in R\} \subseteqq \mathbf{E}^n$$

and given $\mu \in R$, let

$$\mathscr{S}_j^\mu := \{P : \overrightarrow{OP} = \mu\mathbf{e}_j + \mathbf{a};\ \mathbf{a} \in \mathbf{A}_j\} \subseteqq \mathscr{E}^n$$

denote a hyperplane. $\mathscr{S}_j^\mu$ can be considered as an $(n-1)$-dimensional Euclidean space $\mathscr{E}^{n-1}$ with vectors from $\mathbf{A}_j = \mathbf{E}^{n-1}$. The orthonormal basis of $\mathbf{E}^{n-1}$ is $\mathbf{e}_1, \ldots, \mathbf{e}_{j-1}, \mathbf{e}_{j+1}, \ldots \mathbf{e}_n$, and the fixed origin of $\mathscr{E}^{n-1}$ is the point O^μ defined by $\overrightarrow{OO^\mu} = \mu\mathbf{e}_j$. $\mathscr{S}_j^\mu$ bounds two closed semi-spaces in $\mathscr{E}^n$, one of them contains the point $Q \in \mathscr{E}^n$ defined by $\overrightarrow{OQ} = (\mu+1)\mathbf{e}_j$. If $\mathscr{C}_P \cap \mathscr{S}_j^\mu \neq \emptyset$ then $\mathscr{C}_P \cap \mathscr{S}_j^\mu$ is an $(n-1)$-dimensional cube in $\mathscr{E}^{n-1}$ with preferential vertex P^μ, where $\overrightarrow{PP^\mu}$ is a multiple of e_j.

LEMMA 1. *If Ω is a cube mosaic in $\mathscr{E}^n$ and*

$$\Omega_j^\mu := \{\mathscr{C}_{P^\mu}^\mu : \mathscr{C}_{P^\mu}^\mu = \mathscr{C}_P \cap \mathscr{S}_j^\mu,\ \mathscr{C}_P \cap \operatorname{int} \mathscr{F}_j^\mu \neq \emptyset,\ \mathscr{C}_P \in \Omega\}$$

its intersection by a hyperplane $\mathscr{S}_j^\mu$, then Ω_j^μ is a cube mosaic in $\mathscr{E}^{n-1} = \mathscr{S}_j^\mu$ as well.

PROOF. We take the cubes $\mathscr{C}_{P^\mu}^\mu, \mathscr{C}_{Q^\mu}^\mu \in \Omega_j^\mu$ and $\mathscr{C}_P\ \mathscr{C}_Q \in \Omega$ where

$$\overrightarrow{OP} = \overrightarrow{OO^\mu} + \overrightarrow{O^\mu P^\mu} + \overrightarrow{P^\mu P} = \mu\mathbf{e}_j + \overrightarrow{O^\mu P^\mu} + v\mathbf{e}_j, \quad 0 \leqq v < 1$$

$$\overrightarrow{OQ} = \mu\mathbf{e}_j + \overrightarrow{O^\mu Q^\mu} + w\mathbf{e}_j, \quad 0 \leqq w < 1$$

1. If $\operatorname{int} \mathscr{C}_{P^\mu}^\mu \cap \operatorname{int} \mathscr{C}_{Q^\mu}^\mu \neq \emptyset$ and

$$\overrightarrow{P^\mu Q^\mu} = p_1\mathbf{e}_1 + \ldots + p_{j-1}\mathbf{e}_{j-1} + p_{j+1}\mathbf{e}_{j+1} + \ldots + p_n\mathbf{e}_n$$

then because of $|p_1| < 1, \ldots, |p_{j-1}| < 1,\ |p_{j+1}| < 1, \ldots, |p'_n| < 1$;

$$\overrightarrow{PQ} = \overrightarrow{PO} + \overrightarrow{OQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \overrightarrow{P^\mu O^\mu} + (w-v)\,\mathbf{e}_j =$$

$$= p_1\mathbf{e}_1 + \ldots + p_{j-1}\mathbf{e}_{j-1} + (w-v)\mathbf{e}_j + p_{j+1}\mathbf{e}_{j+1} + \ldots + p_n\mathbf{e}_n;\ |w-v| < 1;$$

we have $\operatorname{int} \mathscr{C}_P \cap \operatorname{int} \mathscr{C}_Q \neq \emptyset$, since Ω is a mosaic.

2. Let $P \in \mathscr{E}^{n-1} = \mathscr{S}_j^\mu$, $\{P_1, \ldots, P_r, \ldots\} \subseteqq \operatorname{int} \mathscr{F}_j^\mu$ hold and assume $P_1, P_2, \ldots, P_r, \ldots \ldots \to P$. Ω being a mosaic, this bounded sequence is contained in finitely many of its cubes. Thus, there exists a cube $\mathscr{C}$ such that $\mathscr{C} \in \Omega$, $\mathscr{C} \cap \operatorname{int} \mathscr{F}_j^\mu \neq \emptyset$, $P_{i_1}, P_{i_2}, \ldots, P_{i_r}, \ldots \ldots \in \mathscr{C}$. Obviously, $P_{i_1}, P_{i_2}, \ldots, P_{i_r}, \ldots \to P$. $\mathscr{C}$ is a closed set, hence $P \in \mathscr{C}$. Q.E.D.

PROOF OF THEOREM 5. The statement is trivial when $n=1$. Assume $n>1$, and the theorem to be true in $(n-1)$-dimensions. Take $\mu \in R$ and consider Ω_2^μ. According to Lemma 1, Ω_2^μ is a mosaic in $\mathscr{E}^{n-1}$. Recall that $\mathscr{C}_P \gamma \mathscr{C}_Q$ if and only if $\mathscr{C}_{P^\mu}^\mu \gamma \mathscr{C}_{Q^\mu}^\mu$. Thus, if Δ is a γ class of the mosaic Ω, then

$$\Delta_2^\mu := \{\mathscr{C}_{P^\mu}^\mu : \mathscr{C}_{P^\mu}^\mu = \mathscr{C}_P \cap \mathscr{S}_2^\mu,\ \mathscr{C}_P \cap \operatorname{int} \mathscr{F}_2^\mu \neq \emptyset,\ \mathscr{C}_P \in \Delta\}$$

is also a γ class of the mosaic Ω_2^μ. According to the inductive assumption, $\bar{\Delta}_2^\mu$ is an infinite prism along $\mathbf{e}_1$ for $\bar{\Delta}_2^\mu \neq \emptyset$. Since

$$\bar{\Delta} = \bigcup_{\mu \in R} \bar{\Delta}_2^\mu$$

$\bar{\Delta}$ is an infinite prism along $\mathbf{e}_1$. Q.E.D.

IV. δ classes of cross mosaics $(\mathscr{K}_0^n, \mathbf{L})$

A simple consequence of Theorem 5 is

THEOREM 6. *Let $(\mathscr{K}_0^n, \mathbf{L})$ be a cross mosaic in $\mathscr{E}^n$ and θ a δ class of $(\mathscr{K}_0^n, \mathbf{L})$. Then $\bar{\theta}$ is an infinite prism along* $\mathbf{e}_1$.

PROOF. Obviously, if $\mathscr{C}_P \in \mathscr{K}_R^n$ and $\mathscr{C}_Q \in \mathscr{K}_S^n$ then $\mathscr{C}_P \gamma \mathscr{C}_Q$ holds if and only if $\mathscr{K}_R^n \delta \mathscr{K}_S^n$, by definitions. This implies the statement by Theorem 5. Q.E.D.

Let $\mathbf{X}$ denote the α class containing the vector 0 in the lattice $\mathbf{L}$. $\mathbf{X}$ itself forms a subgroup in $\mathbf{L}$ and the cosets of $\mathbf{L}/\mathbf{X}$ are identical to α classes.

THEOREM 7. *If $\mathbf{X}$ is the α class of $\mathbf{L}$ containing the zero vector, then there exists a subspace $\mathbf{A} \subseteqq \mathbf{E}^n$, where $\mathbf{L} \cap \mathbf{A} = \mathbf{X}$.*

PROOF. Since $\mathbf{X}$ is a lattice, therefore, in conformity with Theorem 3, there are vectors $\mathbf{l}_1, \ldots, \mathbf{l}_m \in \mathbf{E}^n$ such that

$$\mathbf{X} = \{\mathbf{x} : \mathbf{x} = x_1\mathbf{l}_1 + \ldots + x_m\mathbf{l}_m;\ x_1, \ldots, x_m \in Z\}. \tag{1}$$

We define the set

$$\mathbf{A} := \{\mathbf{a} : \mathbf{a} = a_1\mathbf{l}_1 + \ldots + a_m\mathbf{l}_m;\ a_1, \ldots, a_m \in R\}$$

(it may be $\mathbf{X} = \mathbf{A} = \{\mathbf{o}\}$). Obviously $\mathbf{A}$ is a subspace and $\mathbf{L} \cap \mathbf{A} \supseteqq \mathbf{X}$. Assume $\mathbf{L} \cap \mathbf{A} \neq \mathbf{X}$. Thus, there exists $\mathbf{b}$ such that $\mathbf{b} \in \mathbf{L} \cap \mathbf{A}$ and $\mathbf{b} \notin \mathbf{X}$. Hence, if $b = b_1\mathbf{e}_1 + \ldots + b_n\mathbf{e}_n$, then b_1 is an irrational number. Since $\mathbf{L} \cap \mathbf{A}$ is a lattice, $\mathbf{b} \in \mathbf{L} \cap \mathbf{A}$ implies $u\mathbf{b} \in \mathbf{L} \cap \mathbf{A}$ for all $u \in Z$.

1° For a given $u \in Z$, there exist $\mathbf{f}_u \in \mathbf{X}$ such that $u\mathbf{b} + \mathbf{f}_u = z_1\mathbf{l}_1 + \ldots + z_m\mathbf{l}_m$ and $0 \leqq z_1 < 1, \ldots, 0 \leqq z_m < 1$. In fact, for $u\mathbf{b} = t_1\mathbf{l}_1 + \ldots + t_m\mathbf{l}_m$ and $s_i \leqq t_i < s_i + 1$, $s_i \in Z$ $(i = 1, \ldots, m)$. Then, because of (1), $\mathbf{f}_u = -s_1\mathbf{l}_1 - \ldots - s_m\mathbf{l}_m$ is suitable.

2° For $u, v \in Z$ and $u \neq v$, $u\mathbf{b} + \mathbf{f}_u \neq v\mathbf{b} + \mathbf{f}_v$. Indeed, assume

$$(u-v)\,\mathbf{b} + \mathbf{f}_u - \mathbf{f}_v = \mathbf{0}. \tag{2}$$

If $\mathbf{f}_u - \mathbf{f}_v = r_1\mathbf{e}_1 + \ldots + r_n\mathbf{e}_n$ then, because of $(\mathbf{f}_u - \mathbf{f}_v) \in \mathbf{X}$, r_1 is a rational number. Then (2) yields $(u-v)b_1 + r_1 = 0$ contradicting b_1 to be irrational. Because of 2° the set $\{P_u : \overrightarrow{OP_u} = u\mathbf{b} + \mathbf{f}_u\} \subseteqq (0, \mathbf{L})$ is an infinite set and because of 1° it is bounded. Hence, it has a cluster point in contradiction with $(0, \mathbf{L})$ being a point lattice. Q.E.D.

By virtue of Theorems 7 and 4, if $\mathbf{X}$ is spanned by $\mathbf{l}_1, \ldots, \mathbf{l}_m$ and $\mathbf{L}$ is n-dimensional, then there are $\mathbf{l}_{m+1}, \ldots, \mathbf{l}_n$ such that $\mathbf{l}_1, \ldots, \mathbf{l}_m, \mathbf{l}_{m+1}, \ldots, \mathbf{l}_n$ span the lattice $\mathbf{L}$.

Elements of $\mathbf{L}/\mathbf{X}$ i.e. α classes of $\mathbf{L}$ are cosets $\mathbf{X} + \mathbf{l}$ where

$$\mathbf{l} \in \{\mathbf{h} : \mathbf{h} = x_{m+1}\mathbf{l}_{m+1} + \ldots + x_n\mathbf{l}_n;\ x_{m+1}, \ldots, x_n \in Z\}.$$

V. Rational lattices

Let $\mathbf{L}\subseteqq\mathbf{E}^n$ be a lattice spanned by $\mathbf{l}_1, \dots, \mathbf{l}_n$ and $\mathbf{l}_i=a_{i1}\mathbf{e}_1+\dots+a_{in}\mathbf{e}_n$, $i=1, \dots, n$. $\mathbf{L}$ is said to be rational if a_{ij} is rational for all $i, j=1, \dots, n$.

THEOREM 8. *If there is a cross mosaic* $(\mathscr{K}_0^n, \mathbf{L})$ *where the lattice* $\mathbf{L}$ *is not contained in the basic lattice* $\mathbf{N}$ *then there exists a cross mosaic* $(\mathscr{K}_0^n, \mathbf{L}')$ *where* $\mathbf{L}'\nsubseteqq\mathbf{N}$ *and* $\mathbf{L}'$ *is a rational lattice.*

PROOF. Assume $\mathbf{L}\subseteqq\mathbf{N}$ and $(\mathscr{K}_0^n, \mathbf{L})$ is a mosaic. With the previous notations, take

$$\mathbf{X}=\{\mathbf{x}:\mathbf{x}=x_1\mathbf{l}_1+\dots+x_m\mathbf{l}_m;\ x_1, \dots, x_m\in Z\}$$

and

$$\mathbf{L}=\{\mathbf{l}:\mathbf{l}=x_1\mathbf{l}_1+\dots+x_n\mathbf{l}_n;\ x_1, \dots, x_n\in Z\}.$$

Assume $\mathbf{L}$ is not rational. Then there is an irrational a_{ij} in $\mathbf{l}_i=a_{i1}\mathbf{e}_1+\dots+a_{in}\mathbf{e}_n$, $i=1, \dots, n$ we may assume a_{n1} to be irrational, yielding also $m<n$. Let us define a linear transformation $\varphi\colon \mathbf{E}^n\to\mathbf{E}^n$ by $\mathbf{l}_1\varphi=\mathbf{l}_1, \dots, \mathbf{l}_{n-1}\varphi=\mathbf{l}_{n-1}$, $\mathbf{l}_n\varphi=\mathbf{l}_n+\left(\frac{1}{t}-a_{n1}\right)\mathbf{e}_n$. $t>0$, $t\in Z$ is selected such that φ is a bijection.

The system $(\mathscr{K}_0^n, \mathbf{L}\varphi)$ is stated to be mosaic. To be illustrative, the transfoirmation φ shifts some δ classes of the mosaic $(\mathscr{K}_0^n, \mathbf{L})$ along $\mathbf{e}_1$ according to Theorem 6. In fact

$$(\mathbf{X}+\mathbf{l})\varphi=(\mathbf{X}+x_1\mathbf{l}_1+\dots+x_n\mathbf{l}_n)\varphi=\mathbf{X}\varphi+x_1\mathbf{l}_1\varphi+\dots+x_n\mathbf{l}_n\varphi=$$

$$=\mathbf{X}+x_1\mathbf{l}_1+\dots+x_{n-1}\mathbf{l}_{n-1}+x_n\mathbf{l}_n+x_n\left(\frac{1}{t}-a_{n1}\right)\mathbf{e}_1=\mathbf{X}+\mathbf{l}+x_n\left(\frac{1}{t}-a_{n1}\right)\mathbf{e}_1.$$

Obviously $\mathbf{L}\varphi\nsubseteqq\mathbf{N}$, namely $\mathbf{l}_n\varphi\notin\mathbf{N}$.

Any irrational number among $a_{11}, \dots, a_{n-1,1}$ may be cancelled in the same way, yielding a mosaic $(\mathscr{K}_0^n, \mathbf{L}^1)$. In the process from $(\mathscr{K}_0^n, \mathbf{L})$ to $(\mathscr{K}_0^n, \mathbf{L}_1)$ only a_{i1}'s could change. Thus, a subsequent consideration of a_{i2}'s will not impair the improved numbers. This way we get the mosaic $(\mathscr{K}_0^n, \mathbf{L}^2)$. Repeating this process, we arrive at $(\mathscr{K}_0^n, \mathbf{L}^3), \dots, (\mathscr{K}_0^n, \mathbf{L}^n)=(\mathscr{K}_0^n, \mathbf{L}')$. Q.E.D.

VI. Group theoretical formulation

The rational lattice-like cross system $(\mathscr{K}_0^n, \mathbf{L})$ will be sliced into rectangular parallelotopes by means of face planes of cubes constituting the crosses. In the resulting lattice-like system of parallelotopes, the following questions are easy to formulate: Is the system $(\mathscr{K}_0^n, \mathbf{L})$ a mosaic, or is the lattice $\mathbf{L}$ a sublattice of the basic lattice $\mathbf{N}$?

Let $\mathbf{L}$ be a rational lattice spanned by $\mathbf{l}_1, \dots, \mathbf{l}_n$, i.e.

$$\mathbf{l}_i=\frac{p_{i1}}{q_{i1}}\mathbf{e}_1+\dots+\frac{p_{in}}{q_{in}}\mathbf{e}_n,\quad p_{ij}, q_{ij}\in Z,\ q_{ij}>0,\ i, j=1, 2, \dots, n.$$

Denote by q_i the least common multiple of $q_{i1}, \dots, q_{in}$. Let $\psi\colon\mathbf{E}^n\to\mathbf{E}^n$ be a linear transformation defined by $\mathbf{m}_i:=\mathbf{e}_i\psi=\frac{1}{q_i}\mathbf{e}_i$, $i=1, \dots, n$. The corresponding parallelotope will be defined as the image of a cube by ψ.

The lattice of parallelotopes can be defined by $\mathbf{M}:=\mathbf{N}\psi$. Obviusly $\mathbf{L}, \mathbf{N}\subseteqq\mathbf{N}\psi=\mathbf{M}$. The preferential vertices of parallelotopes, forming a cube, are described by the vector set

$$\mathbf{A}=\{\mathbf{a}:\mathbf{a}=a_1\mathbf{m}_1+\dots+a_n\mathbf{m}_n;\ a_1, \dots, a_n\in Z;\ 0\leqq a_1<q_1, \dots, 0\leqq a_n<q_n\}\subseteqq\mathbf{E}^n.$$

Let G be an Abelian group, and $S, H_1, \dots, H_t\subseteqq G$. If all $s\in S$ can be represented in the form $s=h_1+h_2+\dots+h_t$ where the h_i's run over the subsets H_i independently, and if $s=h_1'+h_2'+\dots+h_t'$, $h_1'\in H_1, \dots, h_t'\in H_t$ yields $h_1=h_1', \dots, h_t=h_t'$, then $S=H_1+\dots+H_t$ is called a factorization of the subset S by means of the subsets $H_1, \dots, H_t$.

THEOREM 9. $(\mathscr{K}_0^n, \mathbf{L})$ *is a cross mosaic if and only if there exists an Abelian group* G *and its factorization of the form*

$$G=\{0, q_1g_1, \dots, q_ng_n, -q_1g_1, \dots, -q_ng_n\}+$$

$$+\{0, g_1, 2g_1, \dots, (q_1-1)g_1\}+\{0, g_2, 2g_2, \dots, (q_2-1)g_2\}+\dots$$

$$\dots+\{0, g_n, 2g_n, \dots, (q_n-1)g_n\}.$$

Moreover, $\mathbf{L}\subseteqq\mathbf{N}$ *holds if and only if the first subset is not a subgroup of* G.

The proof of Theorem 9 will be prepared by means of two lemmas.

LEMMA 2. *Let* $\mathbf{L}$ *be a rational lattice. Then a system* $(\mathscr{K}_0^n, \mathbf{L})$ *is a cross mosaic if and only if* $\mathbf{M}/\mathbf{L}$ *has a factorization of the form*

$$\mathbf{M}/\mathbf{L}=\{\langle 0\rangle, q_1\langle\mathbf{m}_1\rangle, \dots, q_n\langle\mathbf{m}_n\rangle, -q_1\langle\mathbf{m}_1\rangle, \dots, -q_n\langle\mathbf{m}_n\rangle\}+$$

$$+\{\langle 0\rangle, \langle\mathbf{m}_1\rangle, 2\langle\mathbf{m}_1\rangle, \dots, (q_1-1)\langle\mathbf{m}_1\rangle\}+\dots+\{\langle 0\rangle, \langle\mathbf{m}_n\rangle, 2\langle\mathbf{m}_n\rangle, \dots, (q_n-1)\langle\mathbf{m}_n\rangle\}.$$

PROOF. A cross system $(\mathscr{K}_0^n, \mathbf{L})$ is a mosaic if and only if for all $\mathbf{m}\in\mathbf{M}$ there are unique vectors $\mathbf{l}\in\mathbf{L}$, $\mathbf{k}\in\mathbf{K}^n$, $\mathbf{a}\in\mathbf{A}$ such that $\mathbf{m}=\mathbf{l}+\mathbf{k}+\mathbf{a}$. This means geometrically that the parallelotope defined by $\mathbf{m}$ is covered by a parallelotope belonging to the cube $\mathbf{k}$ of the cross at the lattice vector $\mathbf{l}$. By definition of the set $\mathbf{A}$, $\mathbf{m}$ can be unambiguously written as $\mathbf{m}=\mathbf{l}+\mathbf{k}+a_1\mathbf{m}_1+\dots+a_n\mathbf{m}_n$, $0\leqq a_1<q_1, \dots, 0\leqq a_n<q_n$. Converting this equality to cosets in $\mathbf{M}/\mathbf{L}$, the statement of the lemma follows. Q.E.D.

COROLLARY. *If* $(\mathscr{K}_0^n, \mathbf{L})$ *is a cross mosaic and* $\mathbf{L}$ *is a rational lattice, then*

1. *The subsets* $\{\langle 0\rangle, \langle\mathbf{m}_i\rangle, 2\langle\mathbf{m}_i\rangle, \dots, (q_i-1)\langle\mathbf{m}_i\rangle\}$ *are not subgroups of* $\mathbf{M}/\mathbf{L}$, *for* $q_i>1$;
2. *the quotient group* $\mathbf{M}/\mathbf{L}$ *is finite, since its order is* $|\mathbf{M}/\mathbf{L}|=(2n+1)q_1\dots q_n$.

In fact, a subset in a factorization cannot have equal elements, thus the element $\langle\mathbf{m}_i\rangle$ is of order $|\langle\mathbf{m}_i\rangle|\geqq q_i$, $i=1, \dots, n$. Similarly, the elements $\langle 0\rangle$, $q_1\langle\mathbf{m}_1\rangle, \dots, q_n\langle\mathbf{m}_n\rangle, -q_1\langle\mathbf{m}_1\rangle, \dots, -q_n\langle\mathbf{m}_n\rangle$ are pairwise different, hence $q_i\langle\mathbf{m}_i\rangle\neq\langle 0\rangle$, $i=1, \dots, n$. Thus $|\langle\mathbf{m}_i\rangle|>q_i$.

LEMMA 3. *Let* $(\mathscr{K}_0^n, \mathbf{L})$ *be a cross mosaic and* $\mathbf{L}$ *a lattice.* $\mathbf{L}\subseteqq\mathbf{N}$ *if and only if the set*

$$\langle\mathbf{K}^n\rangle := \{\langle\mathbf{0}\rangle, \langle\mathbf{e}_1\rangle, \ldots, \langle\mathbf{e}_n\rangle, -\langle\mathbf{e}_1\rangle, \ldots, -\langle\mathbf{e}_n\rangle\} =$$

$$= \{\langle\mathbf{0}\rangle, q_1\langle\mathbf{m}_1\rangle, \ldots, q_n\langle\mathbf{m}_n\rangle, -q_1\langle\mathbf{m}_1\rangle, \ldots, -q_n\langle\mathbf{m}_n\rangle\}$$

is a subgroup of $\mathbf{M}|\mathbf{L}$.

PROOF. Assume $\mathbf{L}\subseteqq\mathbf{N}$. Consider $\mathbf{k}_1, \mathbf{k}_2\in\mathbf{K}^n$ and $\mathbf{l}\in\mathbf{L}$. Because of the assumption, $-\mathbf{k}_2+\mathbf{l}+\mathbf{k}_1=\mathbf{n}\in\mathbf{N}$. Since $(\mathscr{K}_0^n, \mathbf{L})$ is a mosaic, there are $\mathbf{l}'\in\mathbf{L}, \mathbf{k}\in\mathbf{K}^n$, $\mathbf{a}\in\mathbf{A}$ such that $\mathbf{n}=\mathbf{l}'+\mathbf{k}+\mathbf{a}$. Hence $\mathbf{a}=\mathbf{n}-\mathbf{l}'-\mathbf{k}\in\mathbf{N}$. Thus $\mathbf{a}\in\mathbf{A}\cap\mathbf{N}$. By definition of $\mathbf{A}$ and $\mathbf{N}$, $\mathbf{A}\cap\mathbf{N}=\{\mathbf{0}\}$. Hence $\mathbf{a}=\mathbf{0}$, i.e. $-\mathbf{k}_2+\mathbf{l}+\mathbf{k}_1=\mathbf{l}'+\mathbf{k}$. In this equality, turning to cosets in $\mathbf{M}/\mathbf{L}$, we have $-\langle\mathbf{k}_2\rangle+\langle\mathbf{k}_1\rangle=\langle\mathbf{k}\rangle$, ie. $\langle\mathbf{K}^n\rangle$ is a subgroup of $\mathbf{M}/\mathbf{L}$.

Now, assume $\langle\mathbf{K}^n\rangle$ to be a subgroup of $\mathbf{M}/\mathbf{L}$. Take an arbitrary $\mathbf{l}\in\mathbf{L}$. $\mathbf{L}\subseteqq\mathbf{M}$, i.e. $\mathbf{l}\in\mathbf{M}$, implies $\mathbf{l}=x_1\mathbf{m}_1+\ldots+x_n\mathbf{m}_n$, $x_1, \ldots, x_n\in Z$. By means of residual division of $x_i=q_iy_i+z_i$, $0\leqq z_i<q_i$, $i=1, \ldots, n$, $\mathbf{l}=y_1\mathbf{e}_1+\ldots+y_n\mathbf{e}_n+z_1\mathbf{m}_1+\ldots+z_n\mathbf{m}_n$. Converting to cosets in $\mathbf{M}/\mathbf{L}$, we get

$$\langle\mathbf{0}\rangle = y_1\langle\mathbf{e}_1\rangle+\ldots+y_n\langle\mathbf{e}_n\rangle+z_1\langle\mathbf{m}_1\rangle+\ldots+z_n\langle\mathbf{m}_n\rangle.$$

Since $\mathbf{e}_1, \ldots, \mathbf{e}_n\in\mathbf{K}^n$ and since $\langle\mathbf{K}^n\rangle$ is a subgroup of $\mathbf{M}/\mathbf{L}$, there is $\mathbf{k}\in\mathbf{K}^n$ such that $\langle\mathbf{0}\rangle=\langle\mathbf{k}\rangle+z_1\langle\mathbf{m}_1\rangle+\ldots+z_n\langle\mathbf{m}_n\rangle$. The system $(\mathscr{K}_0^n, \mathbf{L})$ is a mosaic, hence, by virtue of the unique representation in the definition of factorization, $\mathbf{k}=\mathbf{0}$ and $z_1=\ldots=z_n=0$, thus $y_1\mathbf{e}_1+\ldots+y_n\mathbf{e}_n=\mathbf{l}\in\mathbf{N}$. Q.E.D.

PROOF OF THEOREM 9. Assume $(\mathscr{K}_0^n, \mathbf{L})$ to be a cross mosaic. According to Theorem 8, let $\mathbf{L}$ be rational. Lemmas 2 and 3 show that $\mathbf{M}/\mathbf{L}$ can be chosen for G.

In turn, assume that an Abelian group G having the required conditions of Theorem 9 exists. As earlier, we define

$$\mathbf{m}_i := \frac{1}{q_i}\mathbf{e}_i,\ i = 1, \ldots, n,$$

$$\mathbf{M} := \{\mathbf{m} : \mathbf{m} = x_1\mathbf{m}_1+\ldots+x_n\mathbf{m}_n;\ x_1, \ldots, x_n\in Z\},$$

$$\mathbf{A} := \{\mathbf{a} : \mathbf{a} = a_1\mathbf{m}_1+\ldots+a_n\mathbf{m}_n;\ a_i\in Z,\ 0\leqq a_i<q_i,\ i = 1, \ldots, n\}.$$

The homomorphism $\Phi\colon \mathbf{M}\to\mathbf{G}$ is defined as

$$(x_1\mathbf{m}_1+\ldots+x_n\mathbf{m}_n)\Phi = x_1g_1+\ldots+x_ng_n, \quad x_1, \ldots, x_n\in Z.$$

The elements $g_1, \ldots, g_n$ are known from the given factorization of the group G. The kernel $\mathbf{L} := \{\mathbf{l} : \mathbf{l}\Phi=0, \mathbf{l}\in\mathbf{M}\}$ of Φ is, in conformity with the homomorphism theorem, a subgroup of $\mathbf{M}$, hence a lattice, and $\mathbf{M}/\mathbf{L}\cong G$. Thus, the given factorization of G serves at the same time that of $\mathbf{M}/\mathbf{L}$. Lemmas 2 and 3 show that $(\mathscr{K}_0^n, \mathbf{L})$ is a cross mosaic and the criterion $\mathbf{L}\subseteqq\mathbf{N}$ holds. Q.E.D.

VII. Group theoretical considerations

Let G be a finite Abelian group.

STATEMENT 1. *If* $g\in G$, $|g|\geqq t$ *and* $t=u\cdot v$, *then*

$$\{0, g, 2g, \ldots, (t-1)g\} = \{0, ug, 2ug, \ldots, (v-1)ug\}+\{0, g, 2g, \ldots, (u-1)g\}$$

is a factorization.

STATEMENT 2. *If* $g_1, g_2, g_3\in G$, $|g_1|\geqq t$, $|g_2|\geqq u$, $|g_3|=v$ *and*

$$\{0, g_1, \ldots, (t-1)g_1\} = \{0, g_2, \ldots, (u-1)g_2\}+\{0, g_3, \ldots, (v-1)g_3\}$$

is a factorization, then $|g_1|=t$.

Only Statement 2 will be proven. Factorization yields

$$\{g_3, g_3+g_1, \ldots, g_3+(t-1)g_1\} = \{0, g_2, \ldots, (u-1)g_2\}+\{g_3, 2g_3, \ldots, vg_3\}.$$

But

$$\{g_3, 2g_3, \ldots, vg_3\} = \{0, g_3, \ldots, (v-1)g_3\}.$$

Thus

$$\{g_3, g_3+g_1, \ldots, g_3+(t-1)g_1\} = \{0, g_1, \ldots, (t-1)g_1\}.$$

Hence

$$\{g_3+g_1, g_3+2g_1, \ldots, g_3+tg_1\} = \{g_1, 2g_1, \ldots, tg_1\}.$$

Finally

$$\{g_3, g_3+g_1, \ldots, g_3+(t-1)g_1\}\cup\{g_1, 2g_1, \ldots, tg_1\} =$$

$$= \{0, g_1, \ldots, (t-1)g_1\}\cup\{g_3+g_1, g_3+2g_1, \ldots, g_3+tg_1\}.$$

We get $\{g_3, tg_1\}=\{0, g_3+tg_1\}$. Since $g_3\neq 0$, $tg_1=0$. Q.E.D.

In a given factorization

$$G = \{0, q_1g_1, \ldots, q_ng_n, -q_1g_1, \ldots, -q_ng_n\}+$$

$$+\{0, g_1, \ldots, (q_1-1)g_1\}+\ldots+\{0, g_n, \ldots, (q_n-1)g_n\}$$

a subset from the last n subsets, having at least two elements, may be further decomposed until the number of its elements is a prime number. None of the resulting subsets will be a subgroup of G, otherwise, according to Statement 2, one subset of the factorization would be a subgroup of G.

VIII. Proof of the main results

PROOF OF THEOREM 1. First, let $2n+1$ be a prime number. Let G be a finite Abelian group and

$$G = \{0, q_1g_1, \ldots, q_ng_n, -q_1g_1, \ldots, -q_ng_n\}+$$

$$+\{0, g_1, \ldots, (q_1-1)g_1\}+\ldots+\{0, g_n, \ldots, (q_n-1)g_n\}$$

a factorization of the group G. By definition, the number of elements in the first subset is a prime number. Omitting eventual subsets of a single zero element from the factorization, also further on, a factorization of G is obtained. In conformity with Statements 1 and 2, the other subsets can be divided into components such that their number of elements is already a prime number, and none of the subsets is a subgroup of G. Obviously, all subsets contain the element 0. According to the following theorem by L. RÉDEI [5], no such decomposition of a finite Abelian group is possible.

RÉDEI'S THEOREM. *If G is a finite Abelian group and $H_1, \dots, H_s \subseteq G$ such that*
1. *$|H_1|, \dots, |H_s|$ are prime numbers,*
2. *$0 \in H_1, \dots, 0 \in H_s$,*
3. *$G = H_1 + \dots + H_s$ is a factorisation of G,*

then one of the subsets $H_1, \dots, H_s$ is a subgroup of G.

Second, let us assume that $2n+1 = u \cdot v$ is not a prime number and $u>1$, $v>1$. The cyclic group C of order $2u \cdot v$ is stated to have a suitable factorization. In fact, denote by g a generator element of C. Obviously

$$C = \left\{-\frac{v-1}{2}g,\ -\frac{v-3}{2}g, \dots, -2g,\ -g,\ 0,\ g,\ 2g, \dots, \frac{v-3}{2}g, \frac{v-1}{2}g\right\} +$$

$$+\{0, vg, 2vg, \dots, (2u-1)vg\}$$

is a factorization, further developed into

$$C = \left\{0, g, 2g, \dots, \frac{v-1}{2}g, -g, -2g, \dots, -\frac{v-1}{2}g\right\} +$$

$$+\{0, 2vg, 4vg, \dots, (2u-2)vg\} + \{0, vg\}.$$

The sum T of the first two subsets will be proved to be a subset of the form $\{0, q_1g_1, \dots, q_ng_n, -q_1g_1, \dots, -q_ng_n\}$ choosing $q_1=2$, $q_2=\dots=q_n=1$. Accordingly, the sets $\{0, vg\}$, $\{0\}$ will act for other sets in the factorization. The subset $\{0, vg\}$ is not a subgroup of C, since $|vg| = 2u > 2$ and $2vg \in T$. The subset T has to be proved not to be a subgroup of C and the elements $T \setminus \{0\}$ can be coupled with their inverses. In fact $\frac{v-1}{2}g \in T$ and $g \in T$, but $\frac{v-1}{2}g + g = -\frac{v-1}{2}g + vg \notin T$ because of the above factorization of C. The described coupling is

$$(j+2kv)g \leftrightarrow [-j+(2u-2k)v]g,$$

$$j = 0, \pm 1, \dots, \pm\frac{v-1}{2};\ k = 0, 1, \dots, u-1;\ (j, k) \neq (0, 0). \quad \text{Q.E.D.}$$

PROOF OF THEOREM 2. The proof will consist in modifying a mosaic of lattice-like crosses. Let $(\mathscr{K}_0^n, \mathbf{L})$ be a mosaic. $2n+1$ is not a prime number, thus in conformity with Theorem 1 it may be assumed that $\mathbf{L} \subsetneqq \mathbf{N}$ and $\mathbf{L}$ is rational. Similarly to the relations $\alpha, \beta, \gamma, \delta$ let us interpret the relations $\alpha_i, \beta_i, \gamma_i, \delta_i;\ i=1, \dots, n$. The vectors $\mathbf{p}, \mathbf{q} \in \mathbf{E}^n$ are said to be in relations α_i, denoted by $\mathbf{p}\alpha_i\mathbf{q}$, if a_i in $\mathbf{q}-\mathbf{p} = a_1\mathbf{e}_1 + \dots + a_n\mathbf{e}_n$ is an integer. δ_i classes of the mosaic $(\mathscr{K}_0^n, \mathbf{L})$ satisfy the statement in Theorem 6: if $(\mathscr{K}_0^n, \mathbf{L})$ is a cross mosaic in the space $\mathscr{E}^n$ and $\theta_i \subseteqq (\mathscr{K}_0^n, \mathbf{L})$ is a δ_i class, then $\bar{\theta}_i$ is an infinite prism along $\mathbf{e}_i$.

α_i classes of the lattice $\mathbf{L}$ are cosets in the quotient group $\mathbf{L}/\mathbf{X}_i$, where $\mathbf{X}_i$ is the α_i class containing the vector $\mathbf{0}$. Clearly $\mathbf{X}_i$ contains vectors distinct from $\mathbf{0}$. $\mathbf{L}$ being rational, α_i classes of the lattice $\mathbf{L}$ are of the form $\mathbf{X}_i + \mathbf{r}$, where $\mathbf{r}$ is rational. Assumption $\mathbf{L} \subsetneqq \mathbf{N}$ leads to the existence of more than one α_i classes for one i if r_i is not an integer in $\mathbf{r} = r_1\mathbf{e}_1 + \dots + r_n\mathbf{e}_n$. Thus, there exist distinct α_i classes $\mathbf{X}_i$ and $\mathbf{X}_i + \mathbf{r}$ $(\mathbf{r} \neq \mathbf{0})$. Let us shift the class $\mathbf{X}_i + \mathbf{r}$ by the vectors $\lambda\mathbf{e}_i$. In order to simplify the proof, let $\lambda = \sqrt{2}$. Now a δ_i class of the cross mosaic $(\mathscr{K}_0^n, \mathbf{L})$ is shifted by the vector $\sqrt{2}\,\mathbf{e}_i$, resulting in a mosaic. The set of vectors $\mathbf{L}^*$ arising by shifting from $\mathbf{L}$ cannot be a lattice. Assume $\mathbf{L}^*$ to be a lattice. Now $j(\mathbf{r}+\sqrt{2}\,\mathbf{e}_i) \in \mathbf{L}^*$ for an arbitrary $j \in Z$, since $(\mathbf{r}+\sqrt{2}\,\mathbf{e}_i) \in \mathbf{L}^*$. Moreover, $\mathbf{X}_i$ is also in $\mathbf{L}^*$. The sets $\mathbf{X}_i + j(\mathbf{r}+\sqrt{2}\,\mathbf{e}_i)$ are pairwise different, namely for $k \neq j$ the vectors $j(\mathbf{r}+\sqrt{2}\,\mathbf{e}_i)$ and $k(\mathbf{r}+\sqrt{2}\,\mathbf{e}_i)$ cannot be α_i related because of the irrationality of $\sqrt{2}$. Thus, if $\mathbf{L}^*$ were lattice-like, then it would have an infinity of equivalence classes resulting from $\mathbf{X}_i$ else than by adding a rational vector. But the construction of $\mathbf{L}^*$ makes it clear that there exists only one such equivalence class. Q.E.D.

IX. An example

The cross $\mathscr{K}_0^7$. Now $2n+1 = 2 \cdot 7 + 1 = 15 = 3 \cdot 5$. Thus $u=3$, $v=5$. Let C be the cyclic group of order $2u \cdot v = 30$. Denote by g a generator element of C. In this special case the sets C, T are the following:

$$C = \{-2g, -g, 0, g, 2g\} + \{0, 5g, 10g, 15g, 20g, 25g\} =$$

$$= \{0, g, 2g, -g, -2g\} + \{0, 10g, 20g\} + \{0, 5g\},$$

$$T = \{0, g, 2g, -g, -2g\} + \{0, 10g, 20g\}.$$

Finally

$$C = \{0, 2 \cdot 5g, g, 2g, 8g, 9g, 11g, 12g,$$

$$-2 \cdot 5g, -g, -2g, -8g, -9g, -11g, -12g\} + \{0, 5g\}.$$

Thus $q_1 = 2$, $q_2 = \dots = q_7 = 1$. We define $\mathbf{m}_i = \frac{1}{q_i}\mathbf{e}_i$, $i = 1, \dots, 7$. Let $\mathbf{M}$ be the lattice spanned by $\mathbf{m}_1, \dots, \mathbf{m}_7$. The homomorphism Φ: $\mathbf{M} \mapsto \mathbf{C}$ is defined as:

$$(x_1\mathbf{m}_1 + \dots + x_7\mathbf{m}_7)\Phi = x_1 5g + x_2 g + x_3 2g + x_4 8g + x_5 9g + x_6 11g + x_7 12g;$$

$$x_1, \dots, x_7 \in Z.$$

The kernel of Φ is $L := \{\mathbf{l} : \mathbf{l}\Phi = 0,\ \mathbf{l} \in \mathbf{M}\}$. The vectors $\mathbf{l}_1, \dots, \mathbf{l}_7$ span the lattice $\mathbf{L}$, where

$$\begin{aligned}
\mathbf{l}_1 &= -\frac{1}{2}\mathbf{e}_1 + 5\mathbf{e}_2 \\
\mathbf{l}_2 &= 30\mathbf{e}_2 \\
\mathbf{l}_3 &= 2\mathbf{e}_2 - \mathbf{e}_3 \\
\mathbf{l}_4 &= 8\mathbf{e}_2 - \mathbf{e}_4 \\
\mathbf{l}_5 &= 9\mathbf{e}_2 - \mathbf{e}_5 \\
\mathbf{l}_6 &= 11\mathbf{e}_2 - \mathbf{e}_6 \\
\mathbf{l}_7 &= 12\mathbf{e}_2 - \mathbf{e}_7 .
\end{aligned}$$

Indeed,

$$\begin{aligned}
\mathbf{l}_1\Phi &= \left(-\frac{1}{2}\mathbf{e}_1 + 5\mathbf{e}_2\right)\Phi = (-\mathbf{m}_1 + 5\mathbf{m}_2)\,\Phi = -5g + 5g = 0 \\
\mathbf{l}_2\Phi &= 30\mathbf{e}_2\Phi = 30\mathbf{m}_2\Phi = 30g = 0 \\
\mathbf{l}_3\Phi &= (2\mathbf{e}_2 - \mathbf{e}_3)\,\Phi = (2\mathbf{m}_2 - \mathbf{m}_3)\,\Phi = 2g - 2g = 0 \\
&\vdots \\
\mathbf{l}_7\Phi &= (12\mathbf{e}_2 - \mathbf{e}_7)\,\Phi = (12\mathbf{m}_2 - \mathbf{m}_7)\,\Phi = 12g - 12g = 0
\end{aligned}$$

and

$$|\mathbf{M}/\mathbf{L}| = \left|\frac{\det(\mathbf{l}_1, \dots, \mathbf{l}_7)}{\det(\mathbf{m}_1, \dots, \mathbf{m}_7)}\right| = |C|.$$

Take the α_1 classes of the lattice $\mathbf{L}$. Let $\mathbf{X}$ be the class containing the vector $\mathbf{0}$. Then

$$\mathbf{X} = \{x_1 2\mathbf{l}_1 + x_2\mathbf{l}_2 + \dots + x_7\mathbf{l}_7;\ x_1, \dots, x_7 \in Z\}.$$

The order of the group $\mathbf{L}/\mathbf{X}$ equals 2. Thus only $\mathbf{X}$, $\mathbf{X}+\mathbf{l}_1$ are α_1 classes of the lattice $\mathbf{L}$.

Now a δ_1 class of the mosaic $(\mathscr{K}_0^7, \mathbf{L})$ is shifted by the vector $\frac{1}{2}\mathbf{e}_1$, resulting a mosaic. The set of vectors $\mathbf{L}^*$ arising by shifting from $\mathbf{L}$ cannot be a lattice. In fact

$$\mathbf{L}^* = \mathbf{X} \cup \left(\mathbf{X} + \mathbf{l}_1 + \frac{1}{2}\mathbf{e}_1\right) = \mathbf{X} \cup (\mathbf{X} + 5\mathbf{e}_2).$$

Assume $\mathbf{L}^*$ to be a lattice. $5\mathbf{e}_2 \in \mathbf{L}^*$ thus $j5\mathbf{e}_2 \in \mathbf{L}^*$ for $j \in Z$. But $10\mathbf{e}_2 \notin \mathbf{L}^*$, because of $10\mathbf{e}_2 \notin \mathbf{X}$ and $10\mathbf{e}_2 \notin \mathbf{X} + 5\mathbf{e}_2$. If the equation

$$10\mathbf{e}_2 = x_2 2\mathbf{l}_1 + x_2\mathbf{l}_1 + \dots + x_7\mathbf{l}_7$$

were satisfied by integers $x_1, \dots, x_7$, then

$$\begin{aligned}
0 &= -x_1 \\
10 &= 5x_1 + 30x_2 + 2x_3 + 8x_4 + 9x_5 + 11x_6 + 12x_7 \\
0 &= -x_3 \\
0 &= -x_4 \\
0 &= -x_5 \\
0 &= -x_6 \\
0 &= -x_7
\end{aligned}$$

finally $10 = 30x_2$. But $x_2 = \frac{1}{3} \notin Z$. The equation $10\mathbf{e}_2 = x_1 2\mathbf{l}_1 + x_2\mathbf{l}_2 + \dots + x_7\mathbf{l}_7 + 5\mathbf{e}_2$ should imply $5 = 30x_2$ that also leads to a contradiction $x_2 = \frac{1}{6} \notin Z$.

Obviusly $\mathbf{L}^* \subseteqq \mathbf{N}$ holds. Thus, there is a non-lattice-like mosaic $(\mathscr{K}_0^7, \mathbf{L}^*)$ of 7-dimensional crosses in which adjacent crosses always join each other along entire cube faces.

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REMARKS ON A PROBLEM OF A. M. BRUCKNER

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1. Let F be a continuous function on $(0, 1)$ and suppose that, for a given sequence $h_n \to 0$, $h_n \neq 0$ the finite limit

$$\lim_{n\to\infty} \frac{F(x+h_n)-F(x)}{h_n} = f(x) \tag{1}$$

exists for every $0<x<1$. In this case F is said to be sq-differentiable and f is its sq-derivative ("sq" stands for "sequential"). An sq-derivative is obviously a Baire 1 function and the problem we refer to in the title is whether every Baire 1 function is the sq-derivative of a suitable F (see [1], pp. 115—117). We answer this question in the negative. It turns out that the property "being sq-differentiable" is a rather restrictive one. For instance, if the sq-derivative f of F is bounded then F is absolutely continuous (Lipschitz 1), and hence $f(x)=F'(x)$ a.e.

NOTATION. For any function F, D^+F, D_+F, D^-F, D_-F denote the usual derived numbers. The Lebesgue measure is denoted by λ. The set A translated by x is denoted by $A+x$.

2. As a starting point we recall a useful theorem on Perron integral (see [2], Theorem (7.3), pp. 204—205). This is

PROPOSITION 1. (i) *Suppose F is a continuous function and g is a Perron integrable function such that $D^+F(x) \geqq g(x)$ a.e. and $D^+F(x) > -\infty$ except at the points of an enumerable set. Then $F(b)-F(a) \geqq (P)\int_a^b g(x)\,dx$.*

(ii) *If $D^+f(x) \geqq g(x) \geqq D_+F(x)$ a.e. and $D^+F(x)>-\infty$, $D_+F(x)<+\infty$ hold everywhere except at the points of an enumerable set, then F is the indefinite Perron integral of g.*

For the applications it should be kept in mind that summable functions as well as derivatives are Perron integrable and their indefinite Perron integral equals the indefinite Lebesgue integral and the primitive, respectively.

COROLLARY 2. *Let f be a sq-derivative of the continuous function F.*

(i) *If $f \geqq 0$, then F is increasing;*

(ii) *if f is summable, then*

$$F(x)=\int_0^x f+\text{const},$$

and hence $F'=f$ a.e. In particular, if f is bounded, then F is Lipschitz 1;

(iii) *if $f(x)=g(x)$ a.e., where g is an ordinary derivative, then $f(x)=g(x)$ everywhere and $F'(x)=f(x)$, i.e. F is a primitive of f;*

(iv) *there always exists an everywhere dense open set U such that $F'(x)=f(x)$ a.e. in U.*

PROOF. Without loss of generality we suppose $h_n>0$ for infinitely many n. Then (1) implies

$$D^+F(x)\geqq f(x)\geqq D_+F(x) \tag{2}$$

for every x. Hence assertions (i)—(iii) are immediate from Proposition 1. Finally, f is a Baire 1 function, thus it is locally bounded on an everywhere dense open set U and (ii) implies (iv).

COROLLARY 3. *Let f and g be the sq-derivatives (with respect to the same sequence $\{h_n\}$) of the continuous functions F and G respectively. If $f(x)=g(x)$ a.e., then $F-G$ is constant.*

PROOF. Obvious by Corollary 2; (iii).

COROLLARY 4. *Let f be a Baire 1 function such that either*

(i) *f is summable but $\left(\int_0^x f(t)\,dt\right)'_{x=x_0}\neq f(x_0)$ at a suitable point x_0, or*

(ii) *there exists a derivative g such that the non-empty set $\{x:f(x)\neq g(x)\}$ is of measure zero. Then f cannot be a sq-derivative. In particular, changing the values of a derivative at finitely many points, the result is a Baire 1 function which is not a sq-derivative.*

EXAMPLE 5. Let f be a bounded function and suppose it is right continuous at every point x. Then taking $F(x)=\int_0^x f$, we have

$$D^+F(x)=D_+F(x)=f(x)$$

at every x. In particular, f is the sq-derivative of F for any positive null-sequence $\{h_n\}$. Taking an increasing function f with jumps, the example above is a non-Darboux sq-derivative.

THEOREM 6. *If the sequence $h_n\to 0$ contains infinitely many positive as well as negative terms then the sq-derivatives with respect to $\{h_n\}$ possess Darboux property.*

PROOF. By the assumption of the theorem $f(x_0)=0$ holds at every local extremum of F where f is the sq-derivative of F. Therefore the standard argument applies.

3. The results so far have been easily obtained. We remark that some of them can also be proved by elementary reasonings, that is without referring to Perron integral.

Now we want to know how sharp are the results above establishing the simple properties of sq-derivatives.

THEOREM 7. *Let a nowhere dense perfect set $P\subset[0,1]$ and a sequence $h_n\to 0$, $h_n\neq 0$ be given. Then there exists a continuous function F such that its sq-derivative (with respect to $\{h_n\}$) exists everywhere, but the ordinary derivative F' fails to exist at the points of P.*

PROOF. The set

$$A=\left[P\cup\bigcup_{n=1}^{\infty}(P+h_n)\right]\cap[0,1]$$

is obviously a nowhere dense perfect set in $[0,1]$. Let $I_1, I_2, \dots$ be the sequence of open intervals contiguous to P, and let δ_k denote the maximal length of intervals contained in $[0,1]\setminus\bigcup_{j=1}^{k-1} I_j$ ($k=1,2,\dots$). Since P is nowhere dense, we have $\lim_{k\to\infty}\delta_k=0$. Let $[a_k,b_k]$ be a closed interval in $I_k\setminus A$ and let F_k ($k=1,2,\dots$) be a non-negative differentiable function on $[a_k,b_k]$ such that

$$F_k(a_k)=F_k'(a_k)=F_k(b_k)=F_k'(b_k)=0, \tag{3}$$

$$\max_{a_k\leqq x\leqq b_k} F_k(x)=\delta_k$$

($k=1,2,\dots$). We define $F(x)$ by

$$F(x)=\begin{cases} F_k(x), & \text{if } x\in[a_k,b_k],\quad k=1,2,\dots,\\ 0 & \text{otherwise.}\end{cases}$$

Then $F(x)$ is continuous on $[0,1]$ and differentiable on $[0,1]\setminus P$. If $x_0\in P$, then $x_0+h_n\in A$ (if only $x_0+h_n\in[0,1]$) and thus

$$\frac{F(x_0+h_n)-F(x_0)}{h_n}=0$$

holds for every n. Hence F is sq-differentiable on $[0,1]$. Furthermore, if $x_0\in P$, then for any fixed N consider the closed component J of $[0,1]\setminus\bigcup_{j=1}^{N} I_j$ containing x_0. Denote by k_N the minimal index with the property $I_{k_N}\subset J$, and let $x_{k_N}\in I_{k_N}$ be a point where F_{k_N} takes its maximum value δ_{k_N}. By the choice of k_N we have $I_k\cap J=\emptyset$ for all $1\leqq k<k_N$. Therefore $J\subset[0,1]\setminus\bigcup_{j=1}^{k_N-1} I_j$, thus $\lambda(J)\leqq\delta_{k_N}$. Hence

$$|x_0-x_{k_N}|\leqq\lambda(J)\leqq\delta_{k_N}$$

and we have

$$\left|\frac{F(x_{k_N})-F(x_0)}{x_{k_N}-x_0}\right|=\left|\frac{\delta_{k_N}}{x_{k_N}-x_0}\right|\geqq 1$$

on the sequence $x_{k_N}\to x_0$. Since

$$\frac{F(x_0+h_n)-F(x_0)}{h_n}=0 \quad (n=1,2,\ldots),$$

the derivative $F'(x_0)$ does not exist, and this completes the proof.

Our next theorem shows that Corollary 2, (iv) has a sharper form if $h_n/h_{n+1}\to 1$.

THEOREM 8. *Let f be the sq-derivative of F with respect to $h_n\to 0$ and suppose $h_n>0$, $\frac{h_n}{h_{n+1}}\to 1$. Then $F'_+(x)=f(x)$ holds on an everywhere dense open set (F'_+ denotes the right hand side derivative of F).*

PROOF. Being a Baire 1 function, f is locally bounded on an everywhere dense open set U. Let $x_0\in U$, and choose $\delta>0$ such that $|f(x)|\leqq M$ holds for $x_0-\delta\leqq x\leqq x_0+\delta$. Let $\varepsilon>0$ be given, then

$$\left|\frac{F(x_0+h_n)-F(x_0)}{h_n}-f(x_0)\right|<\varepsilon \quad\text{and}\quad \left|\frac{h_n}{h_{n+1}}-1\right|<\varepsilon$$

hold for every $n\geqq n_0(\varepsilon)$. Let $0<h<\min(\delta, h_{n_0})$ be arbitrary and denote n the maximal index with $h_n>h$. Then $n\geqq n_0(\varepsilon)$ and $h_{n+1}\leqq h$. By Corollary 2, (ii) we have

$$|F(x_0+h)-F(x_0+h_{n+1})|=\left|\int_{x_0+h_{n+1}}^{x_0+h} f(x)\,dx\right|\leqq M(h-h_{n+1})\leqq M(h_n-h_{n+1}),$$

and hence

$$|F(x_0+h)-F(x_0)-hf(x_0)|\leqq|F(x_0+h)-F(x_0+h_{n+1})|+$$

$$+|F(x_0+h_{n+1})-F(x_0)-h_{n+1}f(x_0)|+|h-h_{n+1}||f(x_0)|\leqq$$

$$\leqq M(h_n-h_{n+1})+\varepsilon h_{n+1}+(h_n-h_{n+1})M=h_{n+1}\left(\varepsilon+2M\left(\frac{h_n}{h_{n+1}}-1\right)\right)<h\varepsilon(2M+1),$$

and this proves $F'_+(x_0)=f(x_0)$.

COROLLARY 9. *If the sequence $h_n\to 0$ contains two subsequences $h_n^{(1)}$, $h_n^{(2)}$ such that $h_n^{(1)}>0$, $h_n^{(2)}<0$ and*

$$\lim\frac{h_n^{(1)}}{h_{n+1}^{(1)}}=\lim\frac{h_n^{(2)}}{h_{n+1}^{(2)}}=1,$$

then $F'(x)=f(x)$ holds on an everywhere dense open set.

Our next theorem points out that Theorem 8 holds no longer true without the restriction $\frac{h_n}{h_{n+1}}\to 1$.

THEOREM 10. *Let $h_1>h_2>\ldots$ be a sequence tending to zero, but $\frac{h_n}{h_{n+1}}\not\to 1$. Let $H=\{r_1, r_2, \ldots\}$ be any countable set in $[0,1]$. Then there exists a continuous function F such that its sq-derivative exists everywhere but the ordinary derivative F' fails to exist at the points of H.*

PROOF. For $0<\alpha<\beta$ we put

$$\varphi_{\alpha,\beta}(x)=\frac{\beta-\alpha}{2\pi}\left[\cos\left(\frac{2\pi}{\beta-\alpha}\left(x-\frac{\alpha+\beta}{2}\right)\right)+1\right].$$

Wse need the following properties of φ:

(4) $$\varphi_{\alpha,\beta}(\alpha)=\varphi'_{\alpha,\beta}(\alpha)=\varphi_{\alpha,\beta}(\beta)=\varphi'_{\alpha,\beta}(\beta)=0,$$

(5) $$\varphi_{\alpha,\beta}\left(\frac{\alpha+\beta}{2}\right)=\frac{\beta-\alpha}{\pi}\quad\text{and}\quad 0\leqq\varphi_{\alpha,\beta}(x)\leqq\frac{\beta-\alpha}{\pi}\quad(\alpha\leqq x\leqq\beta),$$

(6) $$|\varphi_{\alpha,\beta}(x)-\varphi_{\alpha,\beta}(y)|\leqq|x-y|\quad\text{for every } x, y.$$

By the assumption we can find a number $\varepsilon>0$ and a subsequence of integer$_k\{ns\}$ auch that $h_{n+1}/h_n<1-\varepsilon$ for every $n=n_k$. Now we define

$$\psi(x)=\begin{cases}0, & \text{if}\quad x\notin\bigcup\limits_{n=n_k}[h_{n+1}, h_n]\\ \varphi_{\alpha,\beta}(x), & \text{if}\quad x\in[\alpha,\beta]=[h_{n_k+1}, h_{n_k}]\quad\text{for some } k,\end{cases}$$

Ind let

$$F_n(x)=\sum_{p=n+1}^{\infty}\frac{1}{2^p}\psi(x-r_p)\quad(n=0,1,\ldots),\quad F(x)=F_0(x).$$

it is trivial that the function $\psi(x)$ is differentiable everywhere except at $x=0$, where ts sq-derivative with respect to $\{h_n\}$ is 0. Putting

$$t_k=\frac{1}{2}(h_{n_k+1}+h_{n_k})$$

we have by (5)

$$\psi(t_k)=\frac{1}{\pi}(h_{n_k}-h_{n_k+1})=\frac{1}{\pi}h_{n_k}\left(1-\frac{h_{n_k+1}}{h_{n_k}}\right)>\frac{1}{\pi}t_k\varepsilon$$

and hence $D^+\psi(0)\geqq\frac{\varepsilon}{\pi}$. By (6) the estimation $|\psi(x)-\psi(y)|\leqq|y-x|$ obviously holds for every x, y. This implies

$$|F_n(x)-F_n(y)|\leqq\frac{1}{2^n}|x-y|$$

and by a routine argument we get

$$F'_q(x)=\sum_{p=q+1}^{\infty}2^{-p}\psi'(x-r_p)$$

for every $x\neq r_{q+1}, r_{q+2}, \ldots$. In particular, $F'(x)$ exists for every $x\notin H$, and also

$$\gamma_q=\Big(\sum_{p\neq q}2^{-p}\psi(x-r_p)\Big)'$$

exists for $x=r_q$, which shows that the sq-derivative of F in $x=r_q$ is γ_q. Finally

$$D^+F(r_q) \geqq \gamma_q + 2^{-q}\frac{\varepsilon}{\pi} > \gamma_q$$

that is $F'(r_q)$ does not exist, and our proof is complete.

4. The definition of the sq-derivative resembles in many respect to that of the selective derivative introduced by O'Malley ([4], or [1], p. 170). There are many properties possessed by both derivatives. They are, however, very much unlike in the following sense. If F has selective derivative with respect to two selections then the derivatives agree except on a countable set; in addition to this a selective derivative of F is always the approximate derivative of F at almost every point. Our next theorem shows that neither of these properties hold for sq-derivatives.

THEOREM 11. *Let $\varepsilon>0$ be given. Then there exist a perfect set $P\subset[0, 1]$, a function F continuous on $[0, 1]$, and two sequences $h_n\to 0$, $k_n\to 0$ such that*

(i) $\lambda(P)>1-\varepsilon$;

(ii) *the finite limits*

$$f(x) = \lim_{n\to\infty}\frac{F(x+h_n)-F(x)}{h_n}, \quad g(x) = \lim_{n\to\infty}\frac{F(x+k_n)-F(x)}{k_n}$$

exist everywhere;

(iii) $f(x)\neq g(x)$ *for* $x\in P$,

(iv) F'_{app} *does not exist at the density points of P.*

We prepare the proof by the following trivial

LEMMA 12. *Suppose we are given $\varepsilon'>0$, $\eta>0$, a real number α, and a finite system of closed intervals $J_1, J_2, \dots, J_N$ in $[0, 1]$ such that $J_i\cap J_k=\emptyset$ $(i\neq k)$, $\sum_{k=1}^{N}\lambda(J_k)>1-\varepsilon'$. Let $l_1(x), \dots, l_N(x)$ be linear functions defined on $J_1, \dots, J_N$ respectively. Then there exist another system of closed intervals $I_1, \dots, I_M$ and linear functions $L_p(x)=\alpha x+\beta_p$ defined on I_p $(p=1, \dots, M)$ such that $I_p\cap I_q=\emptyset$ $(p\neq q)$, $\sum_{p=1}^{M}\lambda(I_p)>1-\varepsilon'$, $\bigcup_{p=1}^{M} I_p\subset \bigcup_{k=1}^{N} J_k$ and for $x\in I_p\subset J_k$*

$$|L_p(x)-l_k(x)| < \eta$$

holds.

The simple proof of this lemma is omitted.

PROOF OF THE THEOREM. *First step.* Let $J_1^{(1)}=[0, 1]$, $l_1^{(1)}(x)=-x$ and $d_1=1$. Suppose that $n\geqq 1$, and we have defined the pairwise disjoint closed intervals $J_1^{(n)}, \dots, J_{N_n}^{(n)}$ each of which supports a linear function $l_1^{(n)}, \dots, l_{N_n}^{(n)}$ respectively, and a positive number d_n such that $\sum_{k=1}^{N_n}\lambda(J_k^{(n)})>1-\frac{\varepsilon}{2}$. Applying Lemma 12 with $\varepsilon'=\frac{\varepsilon}{2}$, $\eta=2^{-n}d_n$, $\alpha=(-1)^{n+1}$ we obtain the intervals $J_k^{(n+1)}$ and the corresponding linear functions $l_k^{(n+1)}(x)=(-1)^{n+1}x+\beta_k$ $(k=1, \dots, N_{n+1})$ such that

$$|l_k^{(n+1)}(x)-l_j^{(n)}(x)| < 2^{-n}d_n \tag{7}$$

if only $x\in J_k^{(n+1)}\subset J_j^{(n)}$. Furthermore we put

$$\mu = \min\{\lambda(J_k^{(n+1)}):\ k = 1, \dots, N_{n+1}\},$$

$$\nu = \min\{\mathrm{dist}\,(J_k^{(n+1)}, J_j^{(n+1)}):\ 1 \leqq k < j \leqq N_{n+1}\}$$

and $d_{n+1}=\min\left(\frac{1}{n+1}, d_n, \frac{1}{4}\mu, \frac{1}{2}\nu\right)$.

By induction we obtain the sequence $\{H_n, L_n : H_n\to R, d_n\}$ $(n=1, 2, \dots)$ such that

$$H_n = \bigcup_{k=1}^{N_n} J_k^{(n)}, \quad \lambda(H_n) > 1-\frac{\varepsilon}{2}, \tag{8}$$

$$L_n(x) = l_k^{(n)}(x) \quad (x\in J_k^{(n)}), \tag{9}$$

$$\begin{cases} d_n > 0, \ \text{ and } \ x\in H_n,\ x+d_n\in H_n \ \text{ implies} \\ [x, x+d_n] \subset J_k^{(n)} \ \text{ for a suitable } k = 1, \dots, N_n. \end{cases} \tag{10}$$

Therefore

$$\frac{L_n(x+d_n)-L_n(x)}{d_n} = (-1)^n \tag{11}$$

if $x\in H_n$ and $x+d_n\in H_n$ hold simultaneously. We put $Q=\bigcap_{n=1}^{\infty} H_n$. By (8) the set Q is closed and $\lambda(Q)\geqq 1-\frac{\varepsilon}{2}$. According to (7) and (9) we have

$$|L_{n+1}(x)-L_n(x)| \leqq 2^{-n}d_n \leqq 2^{-n} \quad (x\in H_{n+1}) \tag{12}$$

and hence the series

$$L(x) = L_1(x)+\sum_{j=1}^{\infty}(L_{j+1}(x)-L_j(x))$$

is uniformly convergent on Q. The limit function L is continuous on Q and (12) implies

$$|L-L_n| \leqq \sum_{j=n}^{\infty}|L_{j+1}-L_j| \leqq \sum_{j=n}^{\infty}2^{-j}d_j \leqq d_n\sum_{j=n}^{\infty}2^{-j} = 2^{-n+1}d_n \tag{13}$$

as well. The essential property of L we need is

$$\left|\frac{L(x+d_n)-L(x)}{d_n}-(-1)^n\right| \leqq 2^{-n+2} \tag{14}$$

for every n and x, $x+d_n\in Q$, which readily follows from (11) and (13).

Second step. Now we check that our function L cannot have approximate derivative at the density points of Q. Let x_0 be such a point in Q, $x_0\in J_{\nu_n}^{(n)}=[a_n, b_n]$ $(n=1, 2, \dots)$. By the density property

$$\lambda([a_n, b_n]\cap Q) > \frac{7}{8}(b_n-a_n)$$

if n is large enough. If $x_0 \leqq \frac{a_n+b_n}{2}$, then we pick any $y \in Q \cap \left[b_n - \frac{1}{4}(b_n-a_n), b_n\right]$, if $x_0 > \frac{a_n+b_n}{2}$, then we choose any $y \in Q \cap \left[a_n, a_n + \frac{b_n-a_n}{4}\right]$.

In any case

$$\frac{L_n(y)-L_n(x_0)}{y-x_0} = (-1)^n$$

and making use of

$$|y-x_0| \geqq \frac{1}{4}(b_n-a_n) = \frac{1}{4}\lambda(J_{v_n}^{(n)}) \geqq d_n$$

we have by (13)

$$\left|\frac{L(y)-L(x_0)}{y-x_0} - (-1)^n\right| \leqq 2^{-n+2}.$$

The points y considered make up a set of measure at least

$$\frac{1}{4}(b_n-a_n) - \frac{1}{8}(b_n-a_n) = \frac{1}{8}(b_n-a_n)$$

and hence

$$\lim_{x \to x_0} \operatorname{app} \frac{L(x)-L(x_0)}{x-x_0}$$

does not exist.

Since the set P and the function F stated in the theorem are going to be constructed as a subset of positive measure in Q and an extension of $L|_P$, respectively, thus F obviously cannot possess approximate derivative in the density points of P either.

Third step. Now we turn to the construction of P, F, $\{h_n\}$, $\{k_n\}$. We make use of the well known fact that translation is a continuous operation in $\mathcal{L}_p$ spaces ([3], p. 199, Theorem 13.24). In particular, $d_n \to 0$ implies $\lambda(Q \setminus (Q-d_n)) \to 0$ $(n \to \infty)$. Choosing a suitable sequence of even numbers p_n we have

$$\sum_{n=1}^{\infty} \lambda(Q \setminus (Q-d_{p_n})) < \frac{\varepsilon}{4} \tag{15}$$

and similarly

$$\sum_{n=1}^{\infty} \lambda(Q \setminus (Q-d_{q_n})) < \frac{\varepsilon}{4} \tag{16}$$

holds for a suitably rare sequence of odd numbers q_n. We put

$$P = \bigcap_{n=1}^{\infty} \big((Q-d_{p_n}) \cap (Q-d_{q_n}) \cap Q\big),$$

$$h_n = d_{p_n}, \quad k_n = d_{q_n} \quad (n = 1, 2, \ldots).$$

Then $P \subset Q$, and

$$Q \setminus P = \bigcup_{n=1}^{\infty} \big((Q \setminus (Q-d_{p_n})) \cup (Q \setminus (Q-d_{q_n}))\big)$$

and hence by (15) and (16) we have $\lambda(Q \setminus P) < \frac{\varepsilon}{2}$, $\lambda(P) > 1-\varepsilon$, furthermore for any $x \in P$, the relations $x+h_n \in Q$, $x+k_n \in Q$ hold obviously.

Let $\varphi(x) = (\operatorname{dist}(x, P))^2$ $(x \in [0, 1])$. It is easy to see that $\varphi'(x) = 0$ at every $x \in P$. Let $\Phi(x)$ denote the continuous extension of $L(x)$ onto $[0, 1]$ defined linearly on the intervals contiguous to Q. Let $F(x)$ be any function defined on $[0, 1]$ such that F is differentiable on $[0, 1] \setminus P$ and satisfies

$$\Phi(x) \leqq F(x) \leqq \Phi(x) + \varphi(x)$$

all over $[0, 1]$. (F is to be constructed separately on the open intervals contiguous to P as a differentiable function between two given continuous functions $\Phi < \Phi + \varphi$.)

Let $x \in P$. Then $\varphi(x) = 0$, $F(x) = \Phi(x)$ and $x+h_n \in Q$ for every n. Thus

$$\frac{\Phi(x+h_n)-\Phi(x)}{h_n} = \frac{L(x+h_n)-L(x)}{h_n} = 1+o(1),$$

$$\frac{\varphi(x+h_n)-\varphi(x)}{h_n} = o(1),$$

and by

$$\frac{\Phi(x+h_n)-\Phi(x)}{h_n} \leqq \frac{F(x+h_n)-F(x)}{h_n} \leqq \frac{\Phi(x+h_n)-\Phi(x)}{h_n} + \frac{\varphi(x+h_n)}{h_n}$$

we get

$$\lim_{n \to \infty} \frac{F(x+h_n)-F(x)}{h_n} = 1.$$

Similarly

$$\lim_{n \to \infty} \frac{F(x+k_n)-F(x)}{k_n} = -1.$$

This completes the proof of the theorem.

5. Problems. (i) Let the continuous functions F and G be sq-differentiable on the sequences $\{h_n\}$ and $\{k_n\}$, respectively. Suppose that their sq-derivatives agree everywhere. Does $F-G \equiv$ constant follow? Corollaries 2 and 3 imply that $F-G$ is locally constant on an everywhere dense open set.

(ii) What assumptions on $\{h_n\}$ could imply that sq-differentiable functions have approximate derivative almost everywhere?

(iii) Suppose that F is sq-differentiable on $\{h_n\}$ as well as on $\{k_n\}$ and $h_n/k_n \to 1$. Do the sq-derivatives agree almost everywhere?

(iv) Is the class of *all* sq-derivatives additive? uniformly closed? a Borel set in the space of Baire 1 functions (with the topology of uniform convergence)? It is routine from Corollary 2 (ii) that the set of sq-derivatives with respect to a fixed sequence is uniformly closed.

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GROUPS OF BANACH SPACE ISOMETRIES*

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We present several sufficient conditions for the group of isometries of a Banach space to be a Banach—Lie group in the norm topology.

Let $\mathcal{L}(X)$ be the Banach algebra of all bounded linear operators on a Banach space X and $GL(X)$ the group of all invertible elements in $\mathcal{L}(X)$ with the induced topology. We will always assume that any subgroup in $GL(X)$ has the induced topology, too. The group $\mathcal{G}$ of all linear isometries on X is a closed subgroup in $GL(X)$. Thus, if X is finite-dimensional, $\mathcal{G}$ is a Lie group. This is not necessarily true in the infinite dimensional case unless there are additional properties on $\mathcal{G}$ [3]. Here we will essentially require that $\mathcal{G}$ is locally connected by differentiable or rectifiable paths. Thus our results are partial generalizations of the finite dimensional results in Bourbaki [1, Ch. III, § 6, Proposition 4] as well as those of Goto [2].

Let $\mathcal{H}$ be a closed subgroup in $GL(X)$. Denote $\mathcal{H}_I=\{T\in\mathcal{L}(X);\ \exp(sT)\in\mathcal{H}$ if $s\in\mathbf{R}\}$.

Then $\mathcal{H}_I$ is a norm closed Lie algebra with respect to the commutator $[T,S]=TS-ST$. This follows from the standard characterizations of the sum and the commutator. If moreover $\exp(\mathcal{H}_I)$ is a neighbourhood of the identity I, then $\mathcal{H}$ is a real Banach—Lie group with the Lie algebra $\mathcal{H}_I$. This can be proved by the Campbell—Hausdorff—Dynkin theorem. Our first result is

1. Theorem. *Let X be a Banach space (real or complex) and $\mathcal{K}$ a closed bounded subgroup in $GL(X)$ Suppose there exists a neighbourhood $\mathcal{V}$ of the identity I in $\mathcal{K}$ such that we can connect every element in $\mathcal{V}$ to I by a differentiable path in $\mathcal{K}$ contained in the ball of radius* $(1/8)\ln 2$ *around I. Then $\mathcal{K}$ is a real Banach—Lie group.*

We will need some lemmas.

2. Lemma. *Let X be a normed space and $A, B\in\mathcal{L}(X)$. Then for all natural n*

$$\left\|(I+n^{-1}(A+B))^n-(I+n^{-1}A)^n\right\|\leqq\exp(\|A\|)(\exp\|B\|-1).$$

Proof. We have

$$(I+n^{-1}(A+B))^n=(I+n^{-1}A)^n+R,$$

$$\|R\|\leqq\left(1+n^{-1}(\|A\|+\|B\|)\right)^n-(1+n^{-1}\|A\|)^n=$$

$$=(1+n^{-1}\|A\|)^n\left((1+\|B\|(n+\|A\|)^{-1})^n-1\right)\leqq\exp(\|A\|)(\exp\|B\|-1).$$

3. Lemma. *Let X be a Banach space and $\mathcal{H}$ a closed subgroup in $GL(X)$. Let $U(t)$ be a differentiable path in $\mathcal{H}$ and $A(t)=U^{-1}(t)U'(t)$. Then $A(t)\in\mathcal{H}_I$.*

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PROOF. Write $U(t+s)=U(t)(I+sA(t))+o(s)$. Thus $U(t)^{-1}U(t+s)=I+sA(t)+ +B(s)$ where $\lim s^{-1}\|B(s)\|=0\ (s\to 0)$. If we fix s then for any natural n

$$U(t)^{-1}U(t+(s/n)) = I+(s/n)A(t)+B(s/n).$$

Take any $\varepsilon>0$ and let n be so large that $\|B(s/n)\|<\varepsilon(s/n)$. Let $sA(t)=A$ and $nB(s/n)=B$. Clearly $(I+n^{-1}(A+B))^n\in\mathscr{H}$. Since $(I+n^{-1}A)^n\to\exp A$, Lemma 2 implies that for large n $\|(I+n^{-1}(A+B)^n-\exp A\|\leqq\exp(\|A\|)(\exp(\|B\|)-1)+\varepsilon$. But $\|B\|= =\|nB(s/n)\|<\varepsilon s$. Thus every neighbourhood of $\exp A=\exp(sA(t))$ contains an element of $\mathscr{H}$.

4. LEMMA. *Let $\mathscr{H}$ be as in Lemma 3. If $A_0, \dots, A_n\in\mathscr{H}_I$ are such that*

$$\|A_i\| < (1/32)\ln 2$$

and

$$\|\exp(A_0)\dots\exp(A_i)-I\| < (1/4)\ln 2 \quad (i=1, \dots, n) \tag{1}$$

then there exists $B\in\mathscr{H}_I$, $\|B\|<1$, such that

$$\exp(A_0)\dots\exp(A_n) = \exp B.$$

PROOF. We use the Campbell—Hausdorff formula [1, p. 60–63] in the following version: if $U, V\in\mathscr{H}_I$ and $\|U\|+\|V\|<(1/2)\ln 2$ then $h(U, V)=\log(\exp(U)\cdot\exp(V))\in\mathscr{H}_I$ and

$$\|h(U,V)\|\leqq -(1/2)\ln(2-\exp(2(\|U\|+\|V\|))).$$

Let $\exp(A_0)\exp(A_1)=\exp B_1$ where $B_1\in\mathscr{H}_I$. Then by the above formula $\|B_1\|<1$. By (1), $\|\exp(B_1)-I\|<(1/4)\ln 2$. Since

$$B_1 = \log(\exp(B_1)) = \log(I+(\exp(B_1)-I)) =$$
$$= \exp(B_1)-I-(1/2)(\exp(B_1)-I)^2+\dots,$$

we get

$$\|B_1\|\leqq -\ln(1-(1/4)\ln 2) < (3/8)\ln 2.$$

Thus $\|B_1\|+\|A_2\|<(3/8)\ln 2+(1/32)\ln 2<(1/2)\ln 2$ and we can proceed in the same way.

PROOF OF THEOREM 1. We can define a new norm $|\quad|$ on X by $|x|=\sup\{\|Tx\|; T\in\mathscr{K}\}$. Clearly the two norms are equivalent and the same is true for the norms induced on $GL(X)$. All elements in $\mathscr{K}$ are isometries in the new norm. Thus we may assume that $\mathscr{K}$ is a closed subgroup in the group $\mathscr{G}$ of all linear isometries on X.

Let $A\in\mathscr{V}$. There exists a differentiable path $U(t)$ in $\mathscr{K}$ with $U(0)=I$, $U(1)=A$ and such that $\|U(t)-I\|<(1/8)\ln 2$ for all t.

Let $0<\varepsilon<1$. For any t we can find $d(t)>0$ such that for $|h|<d(t)$

$$\|U(t+h)-U(t)-hU'(t)\| < \varepsilon h, \quad \|(U(t+h)-U(t))U^{-1}(t)U'(t)\| < \varepsilon,$$
$$2d(t)\|U'(t)\| < \varepsilon, \quad 2d(t)\|U'(t)\|^2 < \varepsilon.$$

Now we have a covering of $[0, 1]$ by the intervals $(t-d(t), t+d(t))$. Let $t_0, \dots, t_n$ be such that $(t_i-d(t_i), t_i+d(t_i))$ form a subcovering. We may assume that $0= =t_0<t_1<\dots<t_n=1$. For $1\leqq k\leqq n$ choose s_k such that

$$s_k-t_{k-1} < d(t_{k-1}) \quad\text{and}\quad t_k-s_k < d(t_k).$$

Let $s_0=0$, $s_{n+1}=1$. Denote $A(t)=U(t)^{-1}U'(t)$ and $d_k=s_{k+1}-s_k$. Then

$$\|U(s_{k+1})-U(s_k)\exp(d_k A(t_k))\| =$$
$$= \|U(s_{k+1})-U(s_k)-d_k U(s_k)U(t_k)^{-1}U'(t_k)-(1/2!)d_k^2 U(s_k)(U(t_k)^{-1}U'(t_k))^2-\dots\| \leqq$$
$$\leqq \|U(s_{k+1})-U(s_k)-d_k U'(t_k)-d_k(U(s_k)-U(t_k))A(t_k)\|+$$
$$+(1/2!)d_k(d_k\|U'(t_k)\|^2)+(1/3!)d_k(d_k\|U'(t_k)\|^2)(d_k\|U'(t_k)\|)+\dots <$$
$$< \|U(s_{k+1})-U(t_k)-(s_{k+1}-t_k)U'(t_k)\|+$$
$$+\|U(t_k)-U(s_k)-(t_k-s_k)U'(t_k)\|+d_k\varepsilon+(1/2!)d_k\varepsilon+\dots <$$
$$< \varepsilon(s_{k+1}-t_k)+\varepsilon(t_k-s_k)+d_k\varepsilon(1+(1/2!)+(1/3!)+\dots) < 4d_k\varepsilon.$$

This implies that for $k=0, 1, \dots, n$

$$\|U(s_{k+1})-\exp(d_1A(t_1))\dots\exp(d_kA(t_k))\| < 4\varepsilon. \tag{2}$$

Let ε be such that $0<4\varepsilon<(1/8)\ln 2$. Denote $d_kA(t_k)=A_k$. Then

$$\|A_k\| < 2d(t_k)\|U'(t_k)\| < \varepsilon < (1/32)\ln 2.$$

Since

$$\|\exp(A_0)\dots\exp(A_k)-I\| < \|\exp(A_0)\dots\exp(A_k)-U(s_{k+1})\|+\|U(s_{k+1})-I\| <$$
$$< (1/8)\ln 2+(1/8)\ln 2 = (1/4)\ln 2$$

by (1) and $A_r\in\mathscr{K}_I$ by Lemma 3, Lemma 4 implies that

$$\exp(A_0)\dots\exp(A_{n-1}) = \exp B$$

with $B\in\mathscr{K}_I$ and $\|B\|<1$. Let $C=\log A$. By (2), $\|\exp(C)-\exp(B)\|<4\varepsilon$. Since $\mathscr{K}_I$ is closed, $C\in\mathscr{K}_I$. Thus $\exp(\mathscr{K}_I)$ contains $\mathscr{V}$ and the proof is finished.

The idea for the following elegant theorem came from professor Ivan Vidav.

5. THEOREM. *Let X be a Banach space, $\mathscr{G}$ the group of all linear isometries on X and $\mathscr{K}$ a maximal commutative subgroup in $\mathscr{G}$. Suppose there exists a neighbourhood $\mathscr{V}$ of the identity in $\mathscr{K}$ such that for every $U\in\mathscr{V}$ and every $\varepsilon>0$ there exists a finite set, say $\{U_0, \dots, U_n\}\subset\mathscr{K}$, such that $U_0=I$, $U_n=U$, and*

$$\sum\|U_{i+1}-U_i\|^2 < \varepsilon.$$

Then $\mathscr{K}$ is a real Banach—Lie group.

First we present three lemmas stating some simple properties of a Banach algebra A with unit element 1.

6. LEMMA. *Let $a\in A$ and $\|a-1\|<1/2$. Then*

$$\|\log a\|\leqq 2\|a-1\|, \quad \|\log a-(a-1)\|\leqq\|a-1\|^2.$$

Proof. We have

$$\log a = (a-1)-(1/2)(a-1)^2+\dots,$$

$$\|\log a\| \leqq \|a-1\|+\|a-1\|^2+\dots \leqq 2\|a-1\|, \quad \text{etc.}$$

7. **Lemma.** *Let $h\in A$ be such that $\|h\|<1/2$ and $\|\exp h\|=\|\exp(-h)\|=1$. Then $\|1+h\|\leqq 1+\|h\|^2$.*

Proof. We have

$$\|1+h\| = \|\exp(h)\exp(-h)(1+h)\| \leqq \|\exp h\|\,\|\exp(-h)(1+h)\| =$$

$$= \|\exp(-h)(1+h)\| \leqq \|\exp(-h)\|\,\|1+h\| = \|1+h\|.$$

Thus

$$\|1+h\| = \|\exp(-h)(1+h)\| = \left\|(1-h+(1/2)h^2-\dots)(1+h)\right\| =$$

$$= \|1+(1/2)h^2+((1/2!)-(1/3!))h^3+\dots\| \leqq$$

$$\leqq 1+(1/2)\|h\|^2+(1/2)\|h\|^3+\dots \leqq 1+(1/2)\|h\|^2(1-\|h\|^{-1}) \leqq 1+\|h\|^2.$$

An element $a\in A$ is called *skew-symmetric* if $\|\exp(sa)\|=1$ for all real s.

8. **Lemma.** *Let $a\in A$ be such that for every $\varepsilon>0$ there exists a natural n such that*

$$\|1\pm n^{-1}a\| < 1+n^{-1}\varepsilon.$$

Then a is skew-symmetric.

Proof. We observe that $f(t)=\|1+ta\|-1$ is a convex function of the real variable t. Since $f(n^{-1})<\varepsilon n^{-1}$ and $f(0)=0$, $f(t)\leqq\varepsilon t$ for $t\in[0,n^{-1}]$. Similarly, $f(t)\leqq\varepsilon|t|$ for $t\in[-n^{-1},0]$. Thus

$$\|1+ta\| \leqq 1+\varepsilon|t|, \quad t\in[-n^{-1},n^{-1}].$$

Let $s\in\mathbf{R}$ and choose $m\in\mathrm{N}$ such that $m^{-1}|s|<n^{-1}$. Then

$$\left\|(1+(s/m)a)^m\right\| \leqq \left\|1+(s/m)a\right\|^m \leqq (1+\varepsilon(|s|/m))^m \leqq \exp(\varepsilon|s|).$$

Since $(1+(s/m)a)^m$ converges to $\exp(sa)$,

$$\|\exp(sa)\| \leqq \exp(\varepsilon|s|).$$

Since ε was arbitrary, $\|\exp(s)\|\leqq 1$ and the proof is finished.

Proof of Theorem 5. Clearly $\mathscr{K}$ is a closed subgroup in $\mathscr{G}$ and thus in $GL(X)$. We may assume that $\mathscr{V}$ is contained in the open unit ball centered at I. If $U\in\mathscr{V}$ we will show that $A=\log U$ is a skew-symmetric operator which proves the theorem. We fix an $\varepsilon>0$, $\varepsilon<1$. Let $U_0,\dots,U_n$ in $\mathscr{K}$ be such that $U_0=I$, $U_n=U$ and

$$\sum\|U_{i+1}-U_i\|^2 < (1/4)\varepsilon.$$

Lemma 6 implies that

$$\|\log(U_i^{-1}U_{i+1})\| \leqq 2\|U_i^{-1}U_{i+1}-I\| = 2\|U_{i+1}-U_i\|.$$

Since $\mathscr{K}$ is commutative,

$$A=\log U=\sum(\log(U_{i+1})-\log(U_i)) = \sum\log(U_i^{-1}U_{i+1}).$$

Now using Lemma 7 we see that

$$\|I\pm n^{-1}A\| = n^{-1}\|nI\pm A\| = n^{-1}\left\|\sum(I\pm\log(U_i^{-1}U_{i+1}))\right\| \leqq$$

$$\leqq n^{-1}\sum(1+\|\log(U_i^{-1}U_{i+1})\|^2) \leqq n^{-1}\sum(1+4\|U_{i+1}-U_i\|^2) \leqq$$

$$\leqq n^{-1}(n+\varepsilon) = 1+n^{-1}\varepsilon.$$

Thus, by Lemma 8, A is a skew-symmetric operator. Clearly A commutes with all the operators in $\mathscr{K}$, and the same is true for $\exp(tA)$, $t\in\mathbf{R}$. Since $\mathscr{K}$ is a maximal commutative subgroup in $\mathscr{G}$, the one-parameter subgroup $\exp(tA)$ is contained in $\mathscr{K}$. Thus $\mathscr{K}$ is in fact a Lie group.

The non-commutative case of the theorem seems to be more complicated. Let X be a normed space and $f\colon[a,b]\to X$ a function. If $D=\{a=d_0<d_1<\dots<d_n=b\}$ is any partition of the interval $[a,b]$, denote $V_D(f)=\sum\|f(d_i)-f(d_{i-1})\|$ $(d_i\in D)$. If $\sup V_D(f)=V(f,a,b)$ exists and is finite we call it the strong total variation of f on $[a,b]$. If $p\colon[0,1]\to X$ is continuous and $V(p;0,1)<\infty$ we say that p is a rectifiable path.

9. **Theorem.** *Let X be a Banach space and $\mathscr{G}$ the group of all linear isometries on X. Suppose there exists a neighbourhood $\mathscr{V}$ of the identity I in $\mathscr{G}$ such that we can connect any element in $\mathscr{V}$ to I by a rectifiable path in $\mathscr{G}$, contained in the ball of radius* $(1/8)\ln 2$ *around I. Then $\mathscr{G}$ is a real Banach—Lie group.*

Proof. Let $H\in\mathscr{V}$ and let $U(t)$ $(t\in[0,1])$ be a rectifiable path in $\mathscr{G}$ such that $U(0)=I$, $U(1)=H$, and $\|U(t)-I\|<(1/8)\ln 2$ for all t. We will show that $\log H=\log U(1)$ is skew-symmetric.

Let $0\leqq a<b\leqq 1$ and let $D=\{a=t_0<\dots<t_n=b\}$ be a partition of $[a,b]$. We define

$$A_D=\sum_D\log\left(U(t_i)^{-1}U(t_{i+1})\right), \quad B_D=\sum_D U(t_i)^{-1}\left(U(t_{i+1})-U(t_i)\right)$$

and

$$c_D=\sum_D\|U(t_{i+1})-U(t_i)\|^2.$$

We claim that, under the usual partial ordering of partitions $\lim A_D=A(a,b)$ exists and is skew-symmetric.

Since $U(t)$ is continuous, $V(U;a,t)$ is a continuous function of t. For any $\varepsilon>0$ there exists a partition $E=\{e_i\}$ of $[a,b]$ such that $V(U;e_i,e_{i+1})<\varepsilon$. If $E\subset D$,

$$\sum_D\|U(t_{i+1})-U(t_i)\|^2 \leqq \sum_D\varepsilon\|U(t_{i+1})-U(t_i)\| \leqq \varepsilon V(U;a,b).$$

Thus $\lim c_D=0$.

By Lemma 6,

$$\|A_D-B_D\| \leqq \sum_D\|U(t_i)^{-1}U(t_{i+1})-I\|^2 = c_D.$$

Thus the existence of $\lim B_D$ implies the existence of $\lim A_D$ and the equality between them. Moreover we claim that in this case $A(a,b)=\lim A_D$ is skew-symmetric. In fact, let $\varepsilon>0$ and let $A(a,b)=A$. Choose a partition D such that $\|A-A_D\|<(1/2)\varepsilon$ and $c_D<(1/8)\varepsilon$. Lemma 6 implies that

$$\left\|\log\left(U(t_i)^{-1}U(t_{i+1})\right)\right\| \leqq 2\|U(t_i)^{-1}U(t_{i+1})-I\| = 2\|U(t_{i+1})-U(t_i)\|.$$

Thus

$$\sum_D \left\|\log\left(U(t_i)^{-1}U(t_{i+1})\right)\right\|^2 < 4c_D < (1/2)\varepsilon. \tag{3}$$

Let n be the number of points in D. Then using Lemma 7 and (3)

$$\|I \pm n^{-1}A_D\| = n^{-1}\left\|\sum\left(I \pm \log\left(U(t_i)^{-1}U(t_{i+1})\right)\right)\right\| \leqq$$
$$\leqq n^{-1}\sum\left(1+\|\log\left(U(t_i)^{-1}U(t_{i+1})\right)\|^2\right) < 1+(2n)^{-1}\varepsilon.$$

Therefore

$$\|I \pm n^{-1}A\| \leqq \|I \pm n^{-1}A_D\| + n^{-1}\|A-A_D\| < 1+2(1/2n)\varepsilon = 1+n^{-1}\varepsilon.$$

Lemma 8 implies that A is skew-symmetric.

Now we will prove the existence of $\lim B_D$. Suppose we add a point s to D, thus getting a new partition C. Let $t_i < s < t_{i+1}$. Then

$$B_D - B_C = U(t_i)^{-1}U(t_{i+1}) - I - \left(U(t_i)^{-1}U(s) - I\right) -$$
$$-\left(U(s)^{-1}U(t_{i+1}) - I\right) = \left(U(t_i)^{-1} - U(s)^{-1}\right)\left(U(t_{i+1}) - U(s)\right).$$

Now suppose we add several points $s_1 < s_2 < \ldots < s_{n-1}$ between t_i and t_{i+1}. We denote the new partition of $[a, b]$ by F. Let $s_0 = t_i$ and $s_n = t_{i+1}$. Then

$$B_D - B_F = \left(U(s_0)^{-1} - U(s_1)^{-1}\right)\left(U(t_{i+1}) - U(s_1)\right) + \ldots$$
$$\ldots + \left(U(s_{n-2})^{-1} - U(s_{n-1})^{-1}\right)\left(U(t_{i+1}) - U(s_{n-1})\right).$$

Since $\|U(s_i)^{-1} - U(s_{i+1})^{-1}\| = \|U(s_{i+1}) - U(s_i)\|$,

$$\|B_D - B_F\| \leqq \sum \|U(t_{i+1}) - U(s_{i+1})\|\,\|U(s_{i+1}) - U(s_i)\|. \tag{4}$$

U is uniformly continuous on $[a, b]$. For every $\varepsilon > 0$ there exists $\delta > 0$ such that $|t - t'| < \delta$ implies $\|U(t) - U(t')\| < \varepsilon$. If $|t_{i+1} - t_i| < \delta$ then

$$\|B_D - B_F\| < \varepsilon \sum \|U(s_{i+1}) - U(s_i)\| \leqq \varepsilon V(U; s_i, s_{i+1}).$$

Now suppose that partition D is such that $|t_{i+1} - t_i| < \delta$ for all i. If $F \supset D$ then $\|B_D - B_F\| < \varepsilon V(U; a, b)$. This proves that $\lim B_D$ exists.

Denote $M = V(U; 0, 1)$ and let n be a natural number. There exists a partition D of $[0, 1]$, $D = \{0 = w_0 \ldots w_{n-1} = 1\}$ such that $V(U; w_i, w_{i+1}) = n^{-1}M$. By Lemma 6,

$$\left\|\left(\log\left(U(w_i)^{-1}U(w_{i+1})\right) - U(w_i)^{-1}\left(U(w_{i+1}) - U(w_i)\right)\right\| \leqq$$
$$\leqq \|U(w_{i+1}) - U(w_i)\|^2 \leqq \left(V(U; w_i, w_{i+1})\right)^2 = n^{-2}M^2.$$

Now we apply the results of our previous discussions with $a = w_i$ and $b = w_{i+1}$. If $D = \{w_i, w_{i+1}\}$, then $B_D = U(w_i)^{-1}\left(U(w_{i+1}) - U(w_i)\right)$. Since $\lim_E B_E = A(w_i, w_{i+1})$, (4) implies that

$$\left\|A(w_i, w_{i+1}) - U(w_i)^{-1}\left(U(w_{i+1}) - U(w_i)\right)\right\| \leqq$$
$$\leqq \sup\{\|U(w_{i+1}) - U(s)\|;\ s \in [w_i, w_{i+1}]\} \cdot V(U; w_i, w_{i+1}) \leqq$$
$$\leqq \left(V(U; w_i, w_{i+1})\right)^2 = n^{-2}M^2.$$

Using our previous discussion we come to the result

$$\begin{cases} A_i = A(w_i, w_{i+1}) = \log\left(U(w_i)^{-1}U(w_{i+1})\right) + B_i, \\ \qquad\qquad \|B_i\| \leqq 2n^{-2}M^2. \end{cases} \tag{5}$$

Now we need the following estimate:

$$\|U(w_{i+1}) - U(w_i)\exp A_i\| = \|U(w_i)^{-1}U(w_{i+1}) - \exp A_i\| =$$
$$= \left\|\exp\left(\log\left(U(w_i)^{-1}U(w_{i+1})\right)\right) - \exp A_i\right\| =$$
$$= \|\exp(A_i - B_i) - \exp A_i\| \leqq \exp(\|A_i\|)\left(\exp(\|B_i\|) - 1\right)$$

by Lemma 3.

If n is large enough, Lemma 6 implies that

$$\left\|\log\left(U(w_i)^{-1}U(w_{i+1})\right)\right\| \leqq 2\|U(w_i)^{-1}U(w_{i+1}) - I\| =$$
$$= 2\|U(w_{i+1}) - U(w_i)\| \leqq 2V(U: w_i, w_{i+1}) = 2n^{-1}M.$$

Now from (5) we infer that $\|A_i\| < 3n^{-1}M$ if n is large enough. Thus

$$\|U(w_{i+1}) - U(w_i)\exp A_i\| \leqq \exp(3n^{-1}M)\left(\exp(2n^{-2}M^2) - 1\right).$$

If n is large enough, $\exp(3n^{-1}M) < 2$ and $\exp(2n^{-2}M^2) < 1 + 4n^{-2}M^2$. Thus for large n

$$\|U(w_{i+1}) - U(w_i)\exp A_i\| \leqq 8n^{-2}M^2.$$

Finally, in this case

$$\|U(w_{i+1}) - \exp(A_0)\ldots\exp(A_k)\| =$$
$$= \|U(w_{i+1}) - U(w_i)\exp(A_k) + U(w_i)\exp(A_k) - \ldots\| \leqq 8kn^{-2}M^2 \leqq 8n^{-1}M^2.$$

Thus, for large n, we may apply Lemma 4 and write $\exp(A_0)\ldots\exp(A_n) = \exp B$ where B is skew-symmetric and $\|B\| < 1$. The above calculations show that

$$\left\|\exp\left(\log U(1)\right) - \exp B\right\| = \|U(1) - \exp(A_0)\ldots\exp(A_{n-1})\| < 8n^{-1}M^2.$$

Since the space $\mathscr{G}_I$ of all skew-symmetric operators is closed, $\log U(1)$ is skew-symmetric and the proof is finished.

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THE TOPOLOGICAL CHARACTERIZATION OF ALLOWABLE PARTITIONS OF THE CLASS OF UNEQUIVOCAL RINGS

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1. Introduction

In this note we shall work in the variety W of associative or alternative rings. It will be assumed that a class of rings always contains all rings isomorphic to any member of the class. Radical and semisimple classes are meant in the sense of Kurosh and Amitsur and for details of radical theory we refer to [5], [8] and [9].

A class M is said to be hereditary, whenever $I \triangleleft A \in M$ implies $I \in M$. M is closed under subdirect sums if any subdirect sum of M-rings is again in M. Let us recall that every semisimple class is hereditary and subdirectly closed. We shall denote the homomorphic closure and hereditary closure of a class M by $H(M)$ and $J(M)$, respectively, i. e.:

$$H(M) = \{A \in W \mid A \text{ is a homomorphic image of some } M\text{-ring}\} \tag{1}$$

$$J(M) = \{A \in W \mid A \text{ is an accessible subring of some } M\text{-ring}\} \tag{2}$$

B is an accessible subring of a ring A if there is a finite chain $B \triangleleft B_i \triangleleft \dots \triangleleft B_n \triangleleft A$. We shall use the upper radical operator U and semisimple operator S acting on classes as

$$U(M) = \{A \in W \mid H(\{A\}) \cap M \subseteqq \{0\}\} \tag{3}$$

$$S(M) = \{A \in W \mid J(\{A\}) \cap M \subseteqq \{0\}\}. \tag{4}$$

Further, if R is a radical class, then $S(R)$ is a semisimple class; for every semisimple class Q the class $U(Q)$ is always a radical class.

M is a regular class if it satisfies the condition: $H(\{I\}) \cap M \neq 0$, for every $0 \neq I \triangleleft A \in M$. If M is a regular class, then $U(M)$ is a radical class and $M \subseteqq S(U(M))$ holds. It is called the upper radical class determined by M. The terminology upper radical is explained by the fact that if R is a radical class such that each ring from M is in $S(R)$ then the inclusion $R \subseteqq U(M)$ holds. Therefore $S(U(M))$ is the smallest semisimple class containing the regular class M. The semisimple class $S(U(M))$ can be defined as follows

$$S(U(M)) = \{A \in W \mid H(\{I\}) \cap M \neq 0, \text{ for every } 0 \neq I \triangleleft A\}. \tag{5}$$

Let N be an arbitrary class of rings, then the hereditary closure $J(N)$ of N is a regular class. If a semisimple class Q contains the class N, then by the hereditariness of Q the inclusion $J(N) \subseteqq Q$ holds. Thus $S(U(J(N)))$ is the smallest semisimple class containing N, and it is called the semisimple closure of N. The radical class $U(J(N))$ is also called the upper radical class determined by N, and we shall write $U(N)$ for $U(J(N))$.

We shall designate as $\mathscr{L}(N)$ the lower radical, that is, the smallest radical class containing N as obtained by the Kurosh construction starting with $H(N)$. In [6] LEAVITT and YU-LEE LEE have shown that $\mathscr{L}(N)$ can be defined as follows

$$\mathscr{L}(N) = \left\{A \in W \mid J\left(\left\{\frac{A}{I}\right\}\right) \cap H(N) \neq 0, \text{ for every } \frac{A}{I} \neq 0\right\}. \tag{6}$$

A nonzero ring A has been called unequivocal by GARDNER [3], if for any radical R, A is either R-radical or R-semisimple. Thus a radical R divides the class of unequivocal rings into two disjoint classes: An upper class consists of all unequivocal R-semisimple rings, and a lower class consists of all unequivocal R-radical rings. But an arbitrary partition of the class of unequivocal rings into two disjoint classes, an upper class and a lower class, may not lead to a radical. For example, if the simple ring A is in the lower class and the ring $A \oplus A$ is in the upper class of the partition, then no radical corresponds to it. A partition (N, M) of the class of all unequivocal rings is said to be allowable [3] if it satisfies the condition: Every ring in M is $\mathscr{L}(N)$-semisimple.

The class of unequivocal rings was studied by DIVINSKY [2]. In this note it is shown that the lower radical operator and semisimple closure operator act on sets of unequivocal rings as the topological closure operator, and the topological characterization of allowable partitions of the class of unequivocal rings is given.

2. The action of the lower radical operator

LEMMA 1. *For every class N a ring A is $\mathscr{L}(N)$-semisimple if and only if $J(\{A\}) \cap H(N) = 0$ holds.*

PROOF. Assume a ring A to be $\mathscr{L}(N)$-semisimple. Since every semisimple class is hereditary, so every aceessible subring of A is $\mathscr{L}(N)$-semisimple. This implies $J(\{A\}) \cap H(N) = 0$.

Conversely, suppose that a ring A satisfies the condition $\mathscr{L}(\{A\}) \cap H(N) = 0$. Assume B to be an $\mathscr{L}(N)$-ideal of A. If $B \neq 0$, then by (6) every nonzero homomorphic image of B contains a nonzero accessible $H(N)$-subring. In particular, B has a nonzero accessible $H(N)$-subring. This implies $J(\{A\}) \cap H(N) \neq 0$, a contradiction. Thus $B = 0$, and the ring A is $\mathscr{L}(N)$-semisimple. The lemma is proved.

LEMMA 2. *For arbitrary radical classes R_1 and R_2 the equality*

$$S(\mathscr{L}(R_1 \cup R_2)) = S(R_1) \cap S(R_2)$$

holds.

PROOF. $R_i \subseteqq \mathscr{L}(R_1 \cup R_2)$ implies $S(\mathscr{L}(R_1 \cup R_2)) \subseteqq S(R_i)$ and thus $S(\mathscr{L}(R_1 \cup R_2)) \subseteqq S(R_1) \cap S(R_2)$. Now, suppose that a ring A is in $S(R_1) \cap S(R_2)$. Since the semisimple classes $S(R_1)$, $S(R_2)$ are hereditary it follows that the ring A has no nonzero accessible R_i-subring. By Lemma 1 the ring A belongs to the class $S(\mathscr{L}(R_1 \cup R_2))$, since the class $R_1 \cup R_2$ is homomorphically closed. Hence $S(R_1) \cap S(R_2) \subseteqq S(\mathscr{L}(R_1 \cup R))$ is valid. The lemma is proved.

LEMMA 3. *For arbitrary classes N_1 and N_2 of rings the equality*

$$\mathscr{L}(N_1 \cup N_2) = \mathscr{L}(\mathscr{L}(N_1) \cup \mathscr{L}(N_2))$$

holds.

PROOF. Since $N_1 \cup N_2 \subseteqq \mathscr{L}(N_1) \cup \mathscr{L}(N_2)$, we have $\mathscr{L}(N_1 \cup N_2) \subseteqq (\mathscr{L}(N_1) \cup \mathscr{L}(N_2))$. On the other hand, $\mathscr{L}(N_i) \subseteqq \mathscr{L}(N_1 \cup N_2)$ implies $\mathscr{L}(N_1) \cup \mathscr{L}(N_2) \subseteqq \mathscr{L}(N_1 \cup N_2)$. By the minimality of the lower radical, $\mathscr{L}(\mathscr{L}(N_1) \cup \mathscr{L}(N_2)) \subseteqq \mathscr{L}(N_1 \cup N_2)$ holds, and the lemma is proved.

Let us define the operator P acting on the set of all subsets of a given set X of rings as follows:

$$P(G) = \begin{cases} \varnothing & \text{when } G = \varnothing \\ X \cap \mathscr{L}(G) & \text{when } G \neq \varnothing \end{cases} \tag{7}$$

for every $G \subseteqq X$, where $\varnothing$ denotes the empty set.

THEOREM 1. *If X is a set of unequivocal rings, then P is a closure operator in X, that is, P satisfies the following conditions:*

1) $P(\varnothing) = \varnothing$;
2) $G \subseteqq P(G)$;
3) $P(P(G)) = P(G)$;
4) $P(G_1 \cup G_2) = P(G_1) \cup P(G_2)$.

PROOF. Conditions 1) and 2) are obvious. It is clear that $\mathscr{L}(X \cap \mathscr{L}(G)) \subseteqq \mathscr{L}(\mathscr{L}(G)) = \mathscr{L}(G)$, and so we have $P(P(G)) = X \cap \mathscr{L}(P(G)) = X \cap \mathscr{L}(X \cap \mathscr{L}(G)) \subseteqq X \cap \mathscr{L}(G) = P(G)$. This and condition 2) imply 3).

By (7) and Lemma 3 we have $P(G_1 \cup G_2) = X \cap \mathscr{L}(G_1 \cup G_2) = X \cap \mathscr{L}(\mathscr{L}(G_1) \cup \mathscr{L}(G_2)) \supseteqq X \cap (\mathscr{L}(G_1) \cup \mathscr{L}(G_2)) = (X \cap \mathscr{L}(G_1)) \cup (X \cap \mathscr{L}(G_2)) = P(G_1) \cup P(G_2)$.

On the other hand, by Lemmas 2 and 3 we obtain $S(\mathscr{L}(G_1 \cup G_2)) = S(\mathscr{L}(\mathscr{L}(G_1) \cup \mathscr{L}(G_2))) = S(\mathscr{L}(G_1)) \cap S(\mathscr{L}(G_2))$. If an X-ring A does not belong to the class $\mathscr{L}(G_1) \cup \mathscr{L}(G_2)$, then A is in the class $S(\mathscr{L}(G_1)) \cap S(\mathscr{L}(G_2)) = S(\mathscr{L}(G_1 \cup G_2))$, since A is an unequivocal ring. Thus the ring A does not belong to the class $\mathscr{L}(G_1 \cup G_2)$. From this it follows that $A \notin P(G_1) \cup P(G_2)$ implies $A \notin P(G_1 \cup G_2)$. Therefore $P(G_1 \cup G_2) = P(G_1) \cup P(G_2)$ holds. The theorem is proved.

From the proof of the theorem, it is easy to see that the operator P always satisfies conditions 1), 2) and 3) for any set of rings. But if X is not a set of unequivocal rings, then condition 4) may not hold. For example: Let G_1 and G_2 be sets of nonzero simple rings. We choose two rings $A_i \in G_i$, such that $A_1 \notin G_2$ and $A_2 \notin G_1$. It is easy to see that the ring $A_1 \oplus A_2$ is not in $\mathscr{L}(G_1)$ and $\mathscr{L}(G_2)$, but it belongs to $\mathscr{L}(G_1 \cup G_2)$, and it is not an unequivocal ring. Consider the set $X = G_1 \cup G_2 \cup \{A_1 \oplus A_2\}$. We have $P(G_1 \cup G_2) = X \cap \mathscr{L}(G_1 \cup G_2) = X$. But $P(G_1) \cup P(G_2) = (X \cap \mathscr{L}(G_1)) \cup (X \cap \mathscr{L}(G_2)) = G_1 \cup G_2 \neq X$.

Now, let X be a set of unequivocal rings, then the operator P defines a topology on X. We shall denote this topology by t.

PROPOSITION 1. *A subset T of a set X of unequivocal rings is t-closed if and only if $T = X \cap R$, for some radical class R.*

PROOF. Let $T \subseteqq X$ be a t-closed subset, then we have $T=P(T)=X \cap \mathscr{L}(T)$. We take $R=\mathscr{L}(T)$.

Conversely, assume that $T=X \cap R$, for some radical class R. Since R is a radical class, and $X \cap R \subseteqq R$, we have $\mathscr{L}(X \cap R) \subseteqq R$. Hence $P(T)=X \cap \mathscr{L}(X \cap R) \subseteqq X \cap R=T$. Thus $P(T)=T$ follows and T is a t-closed subset.

PROPOSITION 2. *A subset G of a set X of unequivocal rings is t-open if and only if $G=X \cap Q$, for some semisimple class Q.*

PROOF. Since X is a set of unequivocal rings, for every radical class R we have $X=(X \cap R) \cup (X \cap S(R))$ and $(X \cap R) \cap (X \cap S(R))=\emptyset$. A subset G of X is t-open if and only if $X \setminus G$ is t-closed. By Proposition 1 $X \setminus G$ is t-closed if and only if $X \setminus G = X \cap R$, for some radical class R. By the above note $X \setminus G = X \cap R$ holds if and only if $G=X \cap S(R)$. This completes the proof.

3. The action of the semisimple closure operator

Next, we consider the action of the semisimple closure operator $M \to \overline{M} = S(U(J(M)))$ on the class of unequivocal rings. By (5) the class $\overline{M}$ can be defined as follows:

$$\overline{M} = \{A \in W \mid H(\{I\}) \cap J(M) \neq 0, \text{ for every } 0 \neq I \lhd A\}. \tag{8}$$

Let us define the operator P^* acting on the set of all subsets of a given set X of rings as follows:

$$P^*(G) = \begin{cases} \emptyset & \text{when } G=\emptyset \\ X \cap \overline{G} & \text{when } G \neq \emptyset \end{cases} \tag{9}$$

for every $G \subseteqq X$.

THEOREM 2. *If X is a set of unequivocal rings, then P^* is a closure operator in X.*

PROOF. Obviously, the condition $P^*(\emptyset)=\emptyset$ and $G \subseteqq P^*(G)$ are satisfied. We establish the condition $P^*(P^*(G))=P^*(G)$. If $G=\emptyset$, it is clear. If $G \neq \emptyset$, then by (9) we have

$$P^*(P^*(G)) = P^*(X \cap \overline{G}) = X \cap (\overline{X \cap \overline{G}}) \subseteqq X \cap \overline{\overline{G}} = X \cap \overline{G} = P^*(G).$$

This implies $P^*(P^*(G))=P^*(G)$. Finally, we establish the condition: For arbitrary subsets G_1 and G_2 of X, the equality $P^*(G_1 \cup G_2)=P^*(G_1) \cup P^*(G_2)$ holds. Clearly, for arbitrary classes N and M the inclusion $\overline{M} \cup \overline{N} \subseteqq \overline{M \cup N}$ always holds. By (9) we have

$$P^*(G_1) \cup P^*(G_2) = (X \cap \overline{G}_1) \cup (X \cap \overline{G}_2) = X \cap (\overline{G}_1 \cup \overline{G}_2) \subseteqq X \cap (\overline{G_1 \cup G_2}) = P^*(G_1 \cup G_2).$$

Now, suppose that an X-ring A does not belong to the class $\overline{G}_1 \cup \overline{G}_2$. Then the ring A is in $U(G_1)$ and $U(G_2)$ since A is an unequivocal ring. Assume that A is in $\overline{G_1 \cup G_2}$. Then by (8) every nonzero ideal of A can be mapped homomorphically onto a nonzero accessible subring of some set in $G_1 \cup G_2$. This implies that the ring A in $U(G_1) \cap U(G_2)$ has a nonzero homomorphic image in $\overline{G}_1$ or $\overline{G}_2$ which is a contradiction. Thus the ring A does not belong to the class $\overline{G_1 \cup G_2}$. From this it follows that the equality $P^*(G_1 \cup G_2)=P^*(G_1) \cup P^*(G_2)$ holds. The theorem is proved.

By manner as in the case of the operator P we can point out that if X is not a set of unequivocal rings, then the additivity of the operator P^* may fall down.

The operator P^* defines a topology on each set X of unequivocal rings. We shall denote this topology by t^*.

PROPOSITION 3. *A subset T of a set X of unequivocal rings is t^*-closed if and only if $T=X \cap Q$, for some semisimple class Q.*

PROOF. It T is a t^*-closed subset, then $T=P^*(T)=X \cap \overline{T}$ holds. We take $Q=\overline{T}$.

Conversely, assume that the subset T can be written in the form $T=X \cap Q$, for some semisimple class Q. By (9) we have $P^*(T)=P^*(X \cap Q)=X \cap (\overline{X \cap Q}) \subseteqq X \cap \overline{Q} = X \cap Q = T$. Thus $P^*(T)=T$, and T is t^*-closed. The assertion is proved.

PROPOSITION 4. *A subset G of a set X of unequivocal rings is t^*-open if and only if $G=X \cap R$, for some radical class R.*

PROOF. This assertion is proved similarly by means of Proposition 2. We can omit it.

Propositions 1 and 4; Propositions 2 and 3 imply the following dual statements:

PROPOSITION 5. *A subset G of a set X of unequivocal rings is t-closed (t-open) if and only if G is t^*-open (t^*-closed).*

4. Characterization of allowable partitions of the class of unequivocal rings

Denote the class of all unequivocal rings by Y. For every cardinal number α, we put:

$$X_\alpha = \{A \in Y \mid \text{the cardinal number of the set } A \text{ is smaller than } \alpha\}. \tag{10}$$

The operators P and P^* define topologies t_α and t_α^* on X_α, respectively. Suppose that α, β are cardinal numbers and $\alpha \leqq \beta$. Let G be a t_α-open subset of X_α, then by Proposition 2 $G=X_\alpha \cap Q$, where Q is some semisimple class. Since $X_\alpha \subseteqq X_\beta$, we have $G=X_\alpha \cap (X_\beta \cap Q)$. $X_\beta \cap Q$ is a t-open subset of X_β. From this it follows that (X_α, t_α) is a subspace of (X_β, t_β). Similarly, (X_α, t_α^*) is a subspace of (X_β, t_β^*).

PROPOSITION 6. *Let α, β be cardinal numbers and $\alpha \leqq \beta$, then the spaces $(X_\alpha; t_\alpha)$ and $(X_\alpha; t_\alpha^*)$ are subspaces of the spaces $(X_\beta; t_\beta)$ and $(X_\beta; t_\beta^*)$, respectively.*

Thus we get chains of topological spaces:

$$\begin{cases} (X_2; t_2) \subset (X_3; t_3) \subset \ldots \subset (X_\alpha; t_\alpha) \subset \ldots \\ (X_2; t_2^*) \subset (X_3; t_3^*) \subset \ldots \subset (X_\alpha; t_\alpha^*) \subset \ldots \end{cases} \tag{11}$$

where the inclusion maps are continuous.

DEFINITION. Let A and B be unequivocal rings, the ring A is called P-adherence (P^*-adherence) of the ring B, if for some X_α in the chain (11) contains A and B, then A is an adherent point of B in the space $(X_\alpha, t_\alpha)/(X_\alpha, t_\alpha^*)$.

THEOREM 3. *Let (N, M) be a partition of the class of unequivocal rings, then the following are equivalent:*

1) *The partition (N, M) is allowable.*
2) *Every M-ring is not P-adherence of any N-ring.*
3) *Every N-ring is not P^*-adherence of any M-ring.*

PROOF. 1) implies 2): Suppose (N, M) is an allowable partition of the class of all unequivocal rings, that is, $M \subseteqq S(\mathscr{L}(N))$. Assume that there is an M-ring A which is P-adherence of some N-ring B. Let X_α be a set in the chain (11) containing A and B. Then the t-neighbourhood $X_\alpha \cap S(U(M))$ of A must contain B. Hence the N-ring B is in $S(U(M)) \subseteqq S(\mathscr{L}(N)$, a contradiction. Thus every M-ring is not P-adherence of any N-ring.

2) implies 3): Suppose that a partition (N, M) of the class of all unequivocal rings satisfies condition 2). Assume than an N-ring A is P^*-adherence of some M-ring B. Let X_α be a set in the chain (11) containing A and B. Then every t^*-neighbourhood of A in the space (X_α, t_α^*) must contain B.

Suppose that $G = X_\alpha \cap Q$ is any t_α-open neighbourhood of B in the space (X_α, t_α), where Q is some semisimple class. If the ring A is not in G, then $X_\alpha \setminus G = X_\alpha \cap U(Q)$ is a neighbourhood of A in the space (X_α, t_α^*). Hence the ring B must be in $X_\alpha \cap$ $\cap U(Q)$, a contradiction, because B is a Q-ring. Thus the ring A belongs to G. This implies that every t_α-neighbourhood of B in the space (X_α, t_α) contains A, and so the M-ring B is P-adherence of the N-ring A. This contradicts condition 2). Therefore, every N-ring is not P^*-adherence of any M-ring.

3) implies 1): Assume that a partition (N, M) of the class of all unequivocal rings satisfies condition 3). Suppose that there is an M-ring A which does not belong to the class $S(\mathscr{L}(N))$. Since the ring A is unequivocal, so it is an $\mathscr{L}(N)$-ring. By (6) the ring A has a nonzero accessible $H(N)$-subring $B = \frac{C}{I}$, where C is some N-ring. Now, let X_α be a set in the chain (11) containing A and C. Suppose that $G = X_\alpha \cap R$, where R is a radical class, is any open neighbourhood of C in the space (X_α, t_α^*). Since the ring C belongs to the class R, and R is homomorphically closed, the ring B is in R. Hence the ring A has a nonzero accessible R-subring B. By the hereditariness of semisimple classes, the ring A is not R-semisimple. Since A is an unequivocal ring, it must be in R. Hence A belongs to the neighbourhood G of C. From this it follows that C is an adherent point of A in the space (X_α, t_α^*). Thus the N-ring C is P^*-adherence of the M-ring A, a contradiction.

Therefore the inclusion $M \subseteqq S(U(N))$ holds, and the partition (N, M) is allowable. The theorem is proved.

5. Remarks

a) If X is a set of simple rings, then it is easy to see that the spaces $(X; t)$ and $(X; t^*)$ are discrete.

b) *In general, the spaces $(X_\alpha; t_\alpha)$ and $(X_\alpha; t_\alpha^*)$ are not Hausdorff and not compact spaces.*

EXAMPLE. We consider the case $\alpha = \mathscr{N}_0$, where $\mathscr{N}_0$ is the countably infinite cardinality. It is easy to see that every open neighbourhood $G = X_{\mathscr{N}_0} \cap Q$ of a simple ring

A in the space $(X_{\mathscr{N}_0}; t_{\mathscr{N}_0})$, where Q is some semisimple class, contains the ring $A \oplus A$. From this it follows that the space $(X_{\mathscr{N}_0}; t_{\mathscr{N}_0})$ is not a Hausdorff space. Similarly, the space $(X_{\mathscr{N}_0}; t_{\mathscr{N}_0}^*)$ is not a Hausdorff space.

Now, we put

$$M_n = \{A \in X_{\mathscr{N}_0} \mid \text{the ring } A \text{ consists of } n \text{ elements}\}$$

for $n = 2, 3, \ldots$.

Then $G_n = X_{\mathscr{N}_0} \cap S(U(M_n))$ is a $t_{\mathscr{N}_0}$-open set containing M_n. Clearly, $\bigcup_{n=2}^{\infty} G_n = X_{\mathscr{N}_0}$ holds. Assume that there exist sets G_{n_i}, $i = 1, 2, \ldots, k$ such that $\bigcup_{i=1}^{k} G_{n_i} = X_{\mathscr{N}_0}$. Let us choose a prime p satisfying: $p > \max\{n_i, i = 1, \ldots, k\}$. Clearly, the simple ring Z_p (the ring of integers modulo p) is in the radical classes $U(M_{n_i})$, for $i = 1, \ldots, k$. This implies that the ring Z_p does not belong to $\bigcup_{i=1}^{k} G_{n_i} = X_{\mathscr{N}_0}$, a contradiction. From this it follows that the covering $\{G_n\}_{n=2}^{\infty}$ does not contain any finite covering. Thus the space $(X_{\mathscr{N}_0}; t_{\mathscr{N}_0})$ is not compact.

Now, we consider the $t_{\mathscr{N}_0}^*$-open sets $G_n^* = X_{\mathscr{N}_0} \cap \mathscr{L}(M_n)$ containing M_n. It is clear that $\bigcup_{n=2}^{\infty} G_n^* = X_{\mathscr{N}_0}$ holds. In a similar manner we can prove that the covering $\{G_n^*\}_{n=2}^{\infty}$ does not contain any finite covering. Thus also the space $(X_{\mathscr{N}_0}, t_{\mathscr{N}_0}^*)$ is not compact.

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QUASI-IDEALS IN RINGS

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By "ring" we shall mean an associative ring which does not necessarily have an identity, and if B and C are non-empty subsets of a ring, BC will denote the set of all finite sums of the form $\sum_{k=1}^{n} b_k c_k$ where $b_1, \ldots, b_n \in B$ and $c_1, \ldots, c_n \in C$. A *quasi-ideal* of a ring A is a subring Q of A such that $AQ \cap QA \subseteq Q$, and a non-zero quasi-ideal Q is *minimal* if the only quasi-ideals of A contained in Q are $\{0\}$ and Q. Left ideals and right ideals are quasi-ideals and the intersection of a family of quasi-ideals is again a quasi-ideal.

In this paper we show that a quasi-ideal Q of a ring A is minimal if and only if any two of its non-zero elements generate the same left ideal and the same right ideal of A. We also prove that the zero ring on the integers cannot be embedded as a minimal quasi-ideal in any ring. Examples are given to show that the product RL of a right ideal R and a left ideal L need not be a quasi-ideal and that certain theorems in [2] concerning minimal quasi-ideals and minimal one-sided ideals in semiprime rings do not hold for arbitrary non semi-prime rings. These results answer questions raised by L. Márki and published in O. STEINFELD's book on quasi-ideals [2].

We begin by establishing some notation. If x is an element of a ring A, A^1x (respectively, xA^1) will denote the left (respectively, right) ideal of A which is generated by x and $\langle x \rangle$ will denote the subring of A which is generated by x. If A is a ring, $A[\lambda]$ will denote the polynomial ring over A in the indeterminate λ and for each positive integer n, $M_n(A)$ will denote the ring of $n \times n$ matrices with entries from A.

Our first proposition answers Problem 6.1 in [2].

PROPOSITION 1. *A quasi-ideal Q in a ring A is minimal if and only if any two of its non-zero elements generate the same left ideal and the some right ideal of A.*

PROOF. Suppose that Q is a minimal quasi-ideal in a ring A. Let $0 \neq x \in Q$. Then $A^1x \cap Q$ is a non-zero quasi-ideal contained in Q and so $A^1x \cap Q = Q$. Thus $Q \subseteq A^1x$. Let $0 \neq y \in Q$. Then $y \in A^1x$ and so $A^1y \subseteq A^1x$. Similarly, $A^1x \subseteq A^1y$ and so $A^1x = A^1y$. Thus any two non-zero elements of A generate the same left (and similarly the same right) ideal of A.

Conversely, suppose that any two non-zero elements of the quasi-ideal Q generate the same left ideal and the same right ideal of A. Let $\bar{Q}$ be a non-zero quasi-ideal of A, $\bar{Q} \subseteq Q$.

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First assume that $A\bar{Q}\cap Q\neq\{0\}$ and that $\bar{Q}A\cap Q\neq\{0\}$. Let $0\neq q\in A\bar{Q}\cap Q$ and $0\neq r\in\bar{Q}A\cap Q$. Then for any $0\neq x\in Q$, $x\in A^1x=A^1q\subseteqq A\bar{Q}$ and $x\in xA^1=rA^1\subseteqq\bar{Q}A$. Thus $x\in A\bar{Q}\cap\bar{Q}A\subseteqq\bar{Q}$ and so $Q=\bar{Q}$.

Now assume that $A\bar{Q}\cap Q=\{0\}$. Let $0\neq y\in\bar{Q}$. Then for any $0\neq x\in Q$, $A^1x=A^1y$ and so $x\in A^1y$. Thus $x=(n+a)y$ for some integer n and some $a\in A$. It follows that $ay=x-ny\in A\bar{Q}\cap Q=\{0\}$. So $x=ny\in\bar{Q}$ and $Q=\bar{Q}$.

A similar argument shows that $Q=\bar{Q}$ if $\bar{Q}A\cap Q=\{0\}$. Thus Q is minimal.

Proposition 2. *If Q is a minimal quasi-ideal of a ring A then Q, considered as an additive group, is either divisible or a p-group for some prime p.*

Proof. Let Q be a minimal quasi-ideal of a ring A and let p be a prime.

The set $T_p=\{a\in A\colon p^na=0$ for some positive integer $n\}$ is an ideal of A. Thus $Q\cap T_p=Q$ or $Q\cap T_p=\{0\}$. If $Q\cap T_p=Q$, then Q is a p-group.

Suppose $Q\cap T_p=\{0\}$. We will show that in this case Q is p-divisible. Let $0\neq x\in Q$. Since $0\neq px\in Q\cap A^1(px)\cap(px)A^1$, $Q\subseteqq A^1(px)\cap(px)A^1$. Thus there are elements $a, b\in A$ and integers m, n such that $x=(m+a)(px)=(px)(n+b)$. It follows that $(1-pm)x=pax$ and $(1-pn)x=pxb$ and so $kx=pa_1x=pxb_1$ where $a_1=(1-pn)a$, $b_1=(1-pm)b$ and $k=(1-pm)(1-pn)$.

If $kx=0$ we are finished because then $x=p(m+n-pmn)x$. So assume that $kx\neq0$. Now, $0\neq kx\in Q\cap A^1(pa_1x)\cap(pxb_1)A^1$ and so $Q\subseteqq A^1(pa_1x)\cap(pxb_1)A^1\subseteqq A(px)\cap(px)A$. So there are elements $a_2, b_2\in A$ such that $x=pa_2x=pxb_2$ from which it follows that $a_2x=(pa_2x)b_2\in AQ\cap QA\subseteqq Q$. Thus $x=pa_2x$ where $a_2x\in Q$ and so Q is p-divisible.

The above result allows us to answer the ring theoretic part of Problem 6.2 in [2] which asks if every zero ring is a minimal quasi-ideal in some ring. For instance, the zero ring on the integers cannot be a minimal quasi-ideal in any ring because it is not a p-group and it is not p-divisible (for any prime p).

The following example provides a ring A with a left ideal L and a right ideal R such that RL is not a quasi-ideal and hence answers Problems 3.1a and 3.1b in [2].

Example 1. Let k be a field and $\varphi\colon k\to k$ any endomorphism which is not the identity. Let $A=k[x;\varphi]$ be the Ore extension of k with respect to φ: the elements of A are polynomials $\alpha_0+\alpha_1x+\ldots+\alpha_nx^n$ with coefficients in k and the multiplication is determined by the distributive law and the requirement that $x\alpha=\varphi(\alpha)x$ for all $\alpha\in k$. Elementary properties of Ore extensions (skew polynomial rings) can be found in [1]. Let $R=(x+x^2)A$ and $L=A(x+x^2)$.

Choose $\alpha\in k$ such that $\varphi(\alpha)\neq\alpha$. Then since φ is one-to-one $\varphi^2(\alpha)\neq\varphi(\alpha)$. Now

$$[\varphi(\alpha)-\varphi^2(\alpha)]^{-1}[(x+x^2)\alpha-\varphi^2(\alpha)(x+x^2)]=$$

$$=[\varphi(\alpha)-\varphi^2(\alpha)]^{-1}[\varphi(\alpha)x+\varphi^2(\alpha)x^2-\varphi^2(\alpha)x-\varphi^2(\alpha)x^2]=$$

$$=[\varphi(\alpha)-\varphi^2(\alpha)]^{-1}\big[(\varphi(\alpha)-\varphi^2(\alpha))x\big]=x.$$

Thus, $x\in A(x+x^2)A$ and so $x^2+x^3\in ARL\cap RLA$. However, all non-zero polynomials in $RL=(x+x^2)A(x+x^2)$ are of degree at least 4 and so $ARL\cap RLA\nsubseteqq RL$. Thus RL is not a quasi-ideal.

Let A be a ring and consider the following three conditions on A.

(M_q) A has a minimal quasi-ideal.
(M_l) A has a minimal left ideal.
(M_r) A has a minimal right ideal.

Corollary 7.5a in [2] asserts that conditions (M_q), (M_l) and (M_r) are equivalent if the ring A is semi-prime. The following three examples show that in general no two of the above conditions imply the third. This answers the ring theoretic part of Problem 7.2 in [2].

Example 2. Let F_2 be the Galois field of two elements and A be the subring of $M_2(F_2[\lambda])$ which is generated by

$$x=\begin{bmatrix}0&1\\0&0\end{bmatrix}\quad\text{and}\quad y=\begin{bmatrix}0&0\\0&\lambda\end{bmatrix}.$$

Since $Ax=\{0\}$, $Q=\{0, x\}$ is a minimal left ideal and a minimal quasi-ideal of A and so A satisfies (M_l) and (M_q).

Every element of A is a finite sum of the form $\sum_{k=0}^{m}\alpha_kxy^k+\sum_{l=1}^{n}\beta_ly^l$ where the $\alpha_k, \beta_l\in F_2$. It follows that if $0\neq a\in A$, then $a\notin aA\neq\{0\}$. Thus A has no minimal right ideals and consequently A does not satisfy (M_r).

Example 3. Consider A^{op}, the opposite ring of the ring A defined in Example 2: the underlying additive group of A^{op} coincides with that of A and the multiplication in A^{op} is determined by the rule xy (in A^{op}) equals yx (in A).

Clearly, A^{op} satisfies (M_r) and (M_q) but not (M_l).

Example 4. Let F be a field of characteristic 0 and T be the subring of $M_3(F[\lambda])$ which is generated by $\{\alpha x\colon\alpha\in F\}\cup\{\beta y\colon\beta\in F\}$ where

$$x=\begin{bmatrix}0&0&1\\0&0&0\\0&0&0\end{bmatrix}\quad\text{and}\quad y=\begin{bmatrix}1&0&0\\0&\lambda&0\\0&0&0\end{bmatrix}.$$

Notice that $x^2=xy=0$ and $yx=x$. Let $A=T\oplus T^{\mathrm{op}}$.

First we show that T satisfies (M_l) but not (M_q). Every element of T is a finite sum of the form $\alpha_0x+\sum_{k=1}^{m}\alpha_ky^k$ where $\alpha_0, \ldots, \alpha_m\in F$. It is clear that $\{\alpha x\colon\alpha\in F\}$ is a minimal left ideal of T and so T satisfies (M_l).

Let Q be a non-zero quasi-ideal of T. If $0\neq\alpha x\in Q$, then $\langle 2\alpha x\rangle$ is a right ideal of T which is properly contained in Q and so Q is not minimal.

If $0\neq q=\sum_{k=1}^{m}\alpha_ky^k\in Q$, then $Tq=\{yf(y)q\colon f(y)\in F[y]\}\subseteqq TQ\cap QT\subseteqq Q$ and $q\notin Tq$, so again Q is not minimal.

Now suppose that $0\neq q=\alpha_0x+\sum_{k=1}^{m}\alpha_ky^k\in Q$ and $\sum_{k=1}^{m}\alpha_ky^k\neq0$. We will show that Q contains a non-zero polynomial in y and so, as shown in the preceding paragraph,

Q is not minimal. If $\sum_{k=1}^{m}\alpha_k\neq 0$, let $\bar{\alpha}=\left(\sum_{k=1}^{m}\alpha_k\right)^{-1}\alpha_0$. Then

$$yq=\alpha_0x+\sum_{k=1}^{m}\alpha_ky^{k+1}\in TQ$$

and

$$q(\bar{\alpha}x+y)=\alpha_0x+\sum_{k=1}^{m}\alpha_ky^{k+1}\in QT.$$

Thus

$$r=\alpha_0x+\sum_{k=1}^{m}\alpha_ky^{k+1}\in TQ\cap QT\subseteqq Q$$

and so $q-r\in Q$ is a non-zero polynomial in y. On the other hand, if $\sum_{k=1}^{m}\alpha_k=0$, then

$$q^2=\left(\sum_{k=1}^{m}\alpha_ky^k\right)\alpha_0x+\left(\sum_{k=1}^{m}\alpha_ky^k\right)^2=\left(\sum_{k=1}^{m}\alpha_kx\right)\alpha_0+\left(\sum_{k=1}^{m}\alpha_ky^k\right)^2=\left(\sum_{k=1}^{m}\alpha_ky^k\right)^2\in Q.$$

We have thus established that T has no minimal quasi-ideals. Since T has a minimal left ideal, T^{op} has a minimal right ideal and so $A=T\oplus T^{\mathrm{op}}$ satisfies (M_l) and (M_r). We now show that A does not satisfy (M_q).

Let Q be a non-zero quasi-ideal of A and $0\neq(q,r)\in Q$, $q=\alpha_0x+\sum_{k=1}^{m}\alpha_ky^k$ and $r=\beta_0x+\sum_{k=1}^{n}\beta_ky^k$.

Suppose that $\sum_{k=1}^{m}\alpha_ky^k\neq 0$. We have established that this implies the existence of elements $a, b\in T$ such that $aq=qb=f(y)$ is a non-zero polynomial in y. Thus, $(f(y),0)=(a,0)(q,r)=(q,r)(b,0)\in AQ\cap QA\subseteqq Q$ and so $Q\cap T\neq\{0\}$. Similarly, if $\sum_{k=1}^{n}\beta_ky^k\neq 0$, $Q\cap T^{\mathrm{op}}\neq\{0\}$. Since T and T^{op} have no minimal quasi-ideals, both of these cases imply that Q is not minimal and we may reduce to the case in which $(\alpha_0x, \beta_0x)\in Q$ and both α_0 and β_0 are non-zero. But then $\bar{Q}=\langle(2\alpha_0x, 2\beta_0x)\rangle\subsetneqq Q$ and $\bar{Q}$ is a quasi-ideal because $A\bar{Q}\subseteqq T$ and $\bar{Q}A\subseteqq T^{\mathrm{op}}$. Thus A does not satisfy (M_q).

Proposition 7.6a in [2] establishes that if Q is a minimal quasi-ideal of a semiprime ring A, then for all $a\in A$ either $Qa=\{0\}$ or Qa is a minimal quasi-ideal of A. The following example shows that this is not the case for arbitrary rings and hence solves the ring theoretic part of Problem 7.3 in [2].

Example 5. Let F be a field of characteristic 0 and A the subring of $M_2(F)$ which is generated by $\{\alpha x\colon \alpha\in F\}\cup\{\beta y\colon \beta\in F\}$ where

$$x=\begin{bmatrix}0&1\\0&0\end{bmatrix}\quad\text{and}\quad y=\begin{bmatrix}1&0\\0&0\end{bmatrix}.$$

Notice that $x^2=xy=0$, $yx=x$ and $y^2=y$.

Let $Q=\{\beta y\colon \beta\in F\}$. If $0\neq\beta_0y\in Q$, then $A(\beta_0y)=Q$ and $(\beta_0y)A=A$, so Q is a minimal quasi-ideal of A. But $Qx=\{\alpha x\colon \alpha\in F\}$ is not minimal because for each $0\neq\alpha_0\in F$, $\langle\alpha_0x\rangle$ is a non-zero right ideal of A which is properly contained in Qx.

The author would like to thank the referee for his helpful comments. In particular, the referee noted (1) that Gy. Pollák has also provided an (unpublished) solution to Problems 3.1a and 3.1b in [2] and (2) that Example 5 above also solves the semigroup part of Problem 7.3 in [2] because Q is a 0-minimal quasi-ideal of the multiplicative semigroup of A.

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A GEOMETRIC DESCRIPTION OF THE INNER APERTURE OF A CONVEX SET

By

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The inner aperture of a convex set was introduced by D. G. Larman in 1974; a few years later, A. Brøndsted used this notion in his work about intersections of translates of convex sets. The purpose of this note is to give a geometric description of this cone in $\mathbf{R}^n$ and afterwards to extend the study of this notion in an arbitrary real vector space.

1. First we work in $\mathbf{R}^n$ and define the *inner aperture* $P(A)$ of a non-empty convex set A as follows:

$$P(A) := \{y\in\mathbf{R}^n \mid \forall x\in\mathbf{R}^n,\ \exists\lambda \geqq 0 : x+\lambda y\in A\}\cup\{0\}.$$

Remark that this definition is given by Brøndsted [4; p. 335], except the fact that we add the origin to $P(A)$. It is not difficult to see that the inner aperture of the closure of a convex set A coincides with $P(A)$; therefore we may only study the inner aperture of closed convex sets.

We shall describe the cone $P(A)$ by using classical notions. Indeed, $P(A)$ is closely related to the well-known asymptotic cone $O(A)$ of A, i.e. [4; p. 335]

$$O(A) := \{y\in\mathbf{R}^n \mid \exists x\in A,\ \forall\lambda \geqq 0 : x+\lambda y\in A\}.$$

In fact, the difference between these two notions consists in some *directions*, i.e. open half-lines emanating from the origin (further, a translate of a direction is called a *ray*). In order to obtain our main statement, we define two other notions: we call a *boundary direction* of A a direction a translate of which is included in the boundary of A, and an *asymptote direction* of A a direction a translate of which is an asymptote ray of A (i.e. a ray $\varrho\subset\mathbf{R}^n$ of A such that $\delta(A, \varrho)=0$ [5; p. 380]).

Theorem 1. *In* $\mathbf{R}^n$, *for a non-empty closed convex set*, $P(A)$ *coincides with* $O(A)\setminus\mathcal{U}$, *where* $\mathcal{U}$ *is the union of all boundary and asymptote directions of* A.

Proof. It is well-known that $P(A)$ is included in $O(A)$ [4; Corollary, p. 338]. We may suppose that A is a body, i.e. the interior of A is not empty.

Moreover, each boundary or asymptote direction ϱ of A is not included in $P(A)$. Indeed, let $u+\varrho$ be a boundary or asymptote ray of A; by a classical separation theorem, it is possible to find a closed halfspace Σ such that $A\subset\Sigma$, $u+\varrho\subset H$, but $A\not\subset H$, where H denotes the boundary of Σ. In these conditions, let x be a point of the interior of A: it is easy to see that the half-line $y+\varrho$ does not meet A for $y=\lambda u+(1-\lambda)x$ when $\lambda>1$, which proves that ϱ is not included in $P(A)$; therefore the inclusion of $P(A)$ in $O(A)\setminus\mathcal{U}$ is true.

Conversely, let ϱ be a direction included in $O(A)\setminus\mathcal{U}$ and suppose ϱ is not in $P(A)$. There exists a point x such that the half-line $x+\varrho$ does not meet A; but $a+\varrho$ is contained in A for an arbitrary point a of A; then there exists a point b of the segment $[x:a]$ such that the ray $b+\varrho$ is an asymptote ray or a boundary ray of A [5; 1.2, p. 380], which is a contradiction.

This statement allows us to see when the inner aperture $P(A)$ coincides with the asymptotic cone $O(A)$: it is necessary and sufficient that the union $\mathcal{U}$ is empty or, equivalently, that the set A is *continuous* (i.e. A is a closed convex set whose support function is continuous [5; p. 380], because Gale and Klee proved that a closed convex set is continuous if and only if it has neither boundary nor asymptote ray [5; 1.3, p. 381]).

Besides, Bourbaki has introduced the *parabolic* sets, i.e. the closed convex sets A such that, for each $z\notin A$, every generant ray of the cone $z+O(A)$ meets A [3; p. 125]. We have already proved that the notions of continuous and parabolic sets are equivalent in $\mathbf{R}^n$ [1; 6.2, p. 437].

Finally, Brøndsted has proved that $O(A)$ coincides with $P(A)$ if and only if $B(A)\setminus\{0\}$ is open, where $B(A)$ is the barrier cone of A, defined as follows [4; pp. 335—338]

$$B(A):=\{y\in\mathbf{R}^n \mid \sup_{x\in A}\langle x, y\rangle <+\infty\}.$$

In summary, we have

Theorem 2. *In $\mathbf{R}^n$, for a non-empty closed convex set A, the following propositions are equivalent:*

(a) $P(A)=O(A)$;

(b) *A is continuousi;*

(c) *A is parabolici;*

(d) $B(A)\setminus\{0\}$ *is open.*

As a corollary of these two results, we can see when $P(A)$ is closed:

Corollary 1. *In $\mathbf{R}^n$, let A be a non-empty closed convex set. If A is continuous, then $P(A)$ is closed; the converse is true if $P(A)\neq\{0\}$.*

Proof. If A is continuous, Theorem 2 asserts that $P(A)$ coincides with $O(A)$, which is closed for any closed convex A.

Conversely, if $P(A)\neq\{0\}$, then the relative interior rel int $O(A)$ of $O(A)$ is included in $P(A)$ [4; Corollary (k), p. 338]. Suppose moreover that $P(A)$ is closed and A is not continuous: there exists a ray $\varrho=]0:u)$ in $\mathcal{U}$. It is clear that ϱ is included in the boundary of the closed cone $O(A)$. Then, for a ray $\varrho'=]0:u')$ of rel int $O(A)$, the convex hull B of $\varrho\cup\varrho'$ is such that $B\setminus\varrho$ is included in rel int $O(A)$, thus in $P(A)$ and, therefore, ϱ is a subset of the closure of $P(A)$, i.e. of $P(A)$, which contradicts Theorem 1.

Remark that it is possible to find closed convex sets A which are not continuous, but for which $P(A)$ is closed and equal to $\{0\}$; an easy example is given by the set $A=\{(x, y)\in\mathbf{R}^2 \mid x\geqq 0,\ 0\leqq y\leqq 1\}$.

2. Our second purpose is to improve our two theorems and some results of Brøndsted [4] by working in an arbitrary real vector space E (whole dimension can be infinite) with convex sets whose affine hull is not necessarily the whole space. Remark that the corresponding reasonings given in $\mathbf{R}^n$ can be adapted in this general case if we use the geometry of arbitrary linear spaces explained for example in [2], especially results of [1] instead of their versions in $\mathbf{R}^n$ [5]. Therefore the proofs are left to the reader.

We shall now adopt the notations and the terminology of [2]; particularly, for a set A of a real vector space E, we shall denote by ${}^1A$, ${}^iA$, ${}^aA$, ${}^bA$, ${}^mA$ and ${}^cA$ the *affine hull*, the *intrinsic core*, the *linear access*, the *algebraic hull*, the *margin* and the *asymptote cone* of A, respectively [2; pp. 1, 2, 3, 24].

For a non-empty convex set A of E, we study the *relative inner aperture cone* $\mathcal{T}(A)$ of A defined as follows

$$\mathcal{T}(A):=\{y\in E \mid \forall x\in{}^1A,\ \exists\lambda\geqq 0: x+\lambda y\in A\}\cup\{0\}.$$

Notice that, for a non-empty convex set A in $\mathbf{R}^n$, $\mathcal{T}(A)$ coincides with $P(A)$ when the affine hull of A is the whole space; but if ${}^1A\neq\mathbf{R}^n$, then $P(A)=\{0\}$ and thus, from a geometric point of view, the study of $P(A)$ is not very interesting in that case, in contrast to the relative notion $\mathcal{T}(A)$.

Here are some primary properties of this notion:

Theorem 3. *For two non-empty convex sets A and B of an arbitrary real vector space E, one has:*

a) *$\mathcal{T}(A)+A=A$, whence $\mathcal{T}(A)\subset C_A$; moreover, if A is algebraically closed, ${}^i(C_A)\neq\emptyset$ and $\mathcal{T}(A)\neq\{0\}$, then ${}^i(C_A)\subset\mathcal{T}(A)\subset C_A$ and thus ${}^b(\mathcal{T}(A))=C_A$;*

b) *$\mathcal{T}(A)\subset\mathcal{T}(B)$ if $A\subset B$ and ${}^1A={}^1B$;*

c) *if the intrinsic core of A is non-empty, then $\mathcal{T}({}^iA)=\mathcal{T}(A)=\mathcal{T}({}^bA)$.*

Note that the last proposition allows us to suppose that the convex set A is *algebraically closed* (i.e. that $A={}^bA$).

In order to describe the relative inner aperture cone of an algebraically closed non-empty convex set A, we give some definitions: a ray $B=x+[0:u)$ is *asymptote* of A, if $B\subset x+C_A$ and $0\in{}^a(A-B)\setminus(A-B)$ [1; p. 434] (in $\mathbf{R}^n$, this definition is equivalent to the notion of Gale—Klee [5]); as in a Euclidean space, an *asymptote direction* of A is an open ray a translate of which is asymptote of A, and a *marginal direction* is an open ray a translate of which is included in ${}^mA$. So we are able to give this generalization of our first result:

Theorem 1′. *In a real vector space E, for a non-empty algebraically closed convex set A, $\mathcal{T}(A)=C_A\setminus\mathcal{V}$, where $\mathcal{V}$ is the union of all marginal and asymptote directions of A.*

As a consequence of this result, we extend Theorem 2 by using some notions studied in [1; pp. 436—437], but with a slight change in their names: by definition, a non-empty algebraically closed convex set A is *relatively continuous* if it has no ray which is asymptote of A or included in ${}^mA$, and A is *relatively parabolic* if, for each point z of ${}^1A\setminus A$, every generant ray of the cone $z+C_A$ meets A (in $\mathbf{R}^n$, these definitions are equivalent to the notions of Gale—Klee and Bourbaki, respectively, when the affine hull of A is the whole space).

Theorem 2′. *In an arbitrary real vector space E, let A be a non-empty algebraically closed convex set. Then the following propositions are equivalent:*

a) $\mathcal{T}(A)=C_A$,

b) *A is relatively continuous.*

If A is also a convex cell (i.e. a convex set whose intrinsic core is non-empty), a) and b) are equivalent to

c) *A is relatively parabolic.*

It is easy to prove that C_A is algebraically closed for an algebraically closed convex set A of E; but the intrinsic core of C_A can be empty if the dimension of the space is infinite; therefore, in the general case we can only adopt the first part of Corollary 1:

Corollary 1′. *In an arbitrary real vector space E, let A be a non-empty algebraically closed convex set. If A is relatively continuous, then $\mathcal{T}(A)$ is algebraically closed.*

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A REPRESENTATION THEORY FOR THE VARIETY GENERATED BY THE TRIANGLE

By

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Let $W_3=\{0,1,2\}$ and define a pseudo-order, $\leqq$, on W_3 by

$$0<1,\quad 1<2,\quad 2<0.$$

(A relation is a *pseudo-order* if it is reflexive and antisymmetric.) Now define binary operations, $\wedge$ and $\vee$, on W_3 by

$$x\wedge y=x\Leftrightarrow x\vee y=y\Leftrightarrow x\leqq y.$$

The relational structure $\langle W_3;\leqq\rangle$ and the algebra $\langle W_3;\wedge,\vee\rangle$ are both referred to as the *triangle*.

The variety, $\mathbf{W}_3$, generated by $\langle W_3;\wedge,\vee\rangle$ was first studied in E. Fried and G. Grätzer [7]; it is the first member of an important chain of varieties of weakly associative lattices (see E. Fried [6]).

Grätzer and Fried introduced $\mathbf{W}_3$ as a natural generalization of the class of distributive lattices and showed that many important universal algebraic properties of distributive lattices extend to $\mathbf{W}_3$. It is therefore reasonable to ask whether there is a topological representation theory for $\mathbf{W}_3$ similar to H. A. Priestley's representation theory for (bounded) distributive lattices [10, 11]. We shall see that the answer is 'Yes'. We add a nullary operation $*$ to the operations on W_3, and then establish a duality between the variety $\mathbf{A}$ generated by $A=\langle W_3;\wedge,\vee,*\rangle$ and a category $\mathbf{X}$ of structured compact spaces. The algebras in $\mathbf{A}$ are pointed $\mathbf{W}_3$-algebras, and the structure on the objects of $\mathbf{X}$ consists of a pseudo-order $\overset{p}{\leqq}$ and a nullary operation $*$.

The only subdirectly irreducible (bounded) distributive lattice is the two-element chain, and consequently every (bounded) distributive lattice L can be embedded into $\mathbf{2}^X$ for some set X, or equivalently, into the lattice $\mathcal{P}(X)$ of all subsets of X. Part of the role of Priestley's duality, or Stone's duality [13] before it, was to specify the set X and to identify those subsets of X which form a lattice isomorphic to X. A similar line of reasoning applies to the variety $\mathbf{A}$ and the algebra A. The three nontrivial subalgebras of A are the only subdirectly irreducibles in $\mathbf{A}$ and hence every $\mathbf{A}$-algebra, B, can be embedded into A^X for some set X. It is easily seen that A^X is isomorphic to the set of 3-block partitions of X (allowing empty blocks) pseudo-ordered by

$$\langle A_0,A_*,A_2\rangle\leqq\langle B_0,B_*,B_2\rangle\quad\text{if and only if}$$

$$A_0\subseteqq B_0\cup B_*,\quad A_*\subseteqq B_*\cup B_2,\quad\text{and}\quad A_2\subseteqq B_2\cup B_0.$$

Our duality nominates a set X, namely the hom-set $\mathbf{A}(B,A)$, and isolates the set of 3-block partitions of X which forms an algebra isomorphic to B, namely the set

$\mathcal{O}(X)$ of $\mathcal{O}$-partitions of X as defined in Definition 2.1. As is usually the case, it is more convenient to work with a hom-set rather than $\mathcal{O}(X)$: we shall endow A with a natural $\mathbf{X}$-structure, and then concentrate on $\mathbf{X}(X, A)$. A bijection between $\mathcal{O}(X)$ and $\mathbf{X}(X, A)$ is established in the third section.

As applications of our representation we describe the congruence lattice of an arbitrary algebra in $\mathbf{W}_3$ and provide alternative proofs for some of the theorems in [7].

1. Preliminaries

For the purposes of this paper, a category of *structured compact spaces* is any category $\mathbf{X}$, which satisfies

(i) for some fixed type τ, the objects of $\mathbf{X}$ are quadruplets $\langle X; F, R, T\rangle$, where $\langle X; F, R\rangle$ is a structure of type τ in the sense of G. GRÄTZER [5, p. 223], and T is a compact topology on X,

(ii) a morphism from an $\mathbf{X}$-object $\langle X_1; F_1, R_1, T_1\rangle$ to an $\mathbf{X}$-object $\langle X_2; F_2, R_2, T_2\rangle$ is any map $\lambda\colon X_1\to X_2$ which is continuous between the topological spaces $\langle X_1; T_1\rangle$, $\langle X_2; T_2\rangle$, and which is a homomorphism from $\langle X_1; F_1, R_1\rangle$ into $\langle X_2; F_2, R_2\rangle$.

Given a category $\mathbf{X}$ of structured compact spaces, unless otherwise indicated we shall make no notational distinction between an $\mathbf{X}$-object and its underlying set. This same convention will apply to all concrete categories, and should cause no problems since in those cases where a set, structured in different ways, lies in different categories $\mathbf{A}$ and $\mathbf{X}$, it will be clear from the given context whether it is acting as an object of $\mathbf{A}$ or an object of $\mathbf{X}$.

Let A be an algebra whose base set is endowed with a compact Hausdorff topology. Then A is a *compact topological algebra* if each operation on A is continuous. $\mathbf{A}$ is the *prevariety* generated by a class $\mathbf{K}$ of algebras if $\mathbf{A}=ISP(\mathbf{K})$. Given categories $\mathbf{A}$ and $\mathbf{X}$, an adjunction $\langle D, E, \eta, \varepsilon\rangle\colon \mathbf{A}\to\mathbf{X}^{\mathrm{op}}$ is a *duality* (between $\mathbf{A}$ and $\mathbf{X}$) if, for each $B\in\mathbf{A}$, $\eta_B\colon B\to ED(B)$ is an $\mathbf{A}$-isomorphism, and is a *full duality* if also, for each $X\in\mathbf{X}$, $\varepsilon_X\colon X\to DE(X)$ is an $\mathbf{X}$-isomorphism.

Following B. A. DAVEY [4], many interesting dualities can be established using the technique set out below.

The category $\mathbf{A}$ is the prevariety generated by a compact topological algebra A, and $\mathbf{X}$ is a category of structured compact spaces. Then for each $B\in\mathbf{A}$, under the topology which it inherits as a closed subspace of A^B, the space $\mathbf{A}(B, A)$ is compact [4, Lemma 2.3]; in this way $\mathbf{A}(-, A)$ becomes a well-defined functor from $\mathbf{A}$ into $\mathbf{Comp}^{\mathrm{op}}$ [4, Proposition 2.4], where, as usual, $\mathbf{Comp}$ denotes the category of compact spaces and continuous maps. Let $|-|\colon \mathbf{X}\to\mathbf{Comp}$ be the forgetful functor, and suppose that $D\colon \mathbf{A}\to\mathbf{X}^{\mathrm{op}}$ is a functor satisfying $|D(-)|=\mathbf{A}(-, A)$.

Thus, for each $B\in\mathbf{A}$, $\mathbf{A}(B, A)$ can be structured so that it lies in $\mathbf{X}$. But by [8, Corollary 1, p. 167], all free algebras exist in $\mathbf{A}$, and if $\{y_\lambda\colon \lambda<\varkappa\}$ freely generates $F_{\mathbf{A}}(\varkappa)$, then the map $\varrho_\varkappa\colon \mathbf{A}(F_{\mathbf{A}}(\varkappa), A)\to A^\varkappa$, given by $g\varrho_\varkappa=\langle x_\lambda g\rangle_{\lambda<\varkappa}$, is a homeomorphism if $A^\varkappa$ has the product topology $T_\varkappa$ [4, Proposition 2.6]. Consequently, for each $\varkappa>0$, $\varrho_\varkappa$ induces a structure on $A^\varkappa$ so that it too lies in $\mathbf{X}$; that is, if $D(F_{\mathbf{A}}(\varkappa))=\langle \mathbf{A}(F_{\mathbf{A}}(\varkappa), A); F, R, T\rangle$ lies in $\mathbf{X}$, then for each m-ary operation f in F, each n-ary relation r in R, one defines an m-ary operation f^1, and an n-ary relation r^1, on $A^\varkappa$ by;

$$\text{for all } a_0, \dots, a_{m-1}, b_0, \dots, b_{n-1}\in A^\varkappa,\quad f^1(a_0, \dots, a_{m-1}) = f(a_0\varrho_\varkappa^{-1}, \dots, a_{m-1}\varrho_\varkappa^{-1})\varrho_\varkappa,$$

and

$$r^1(b_0, \dots, b_{n-1}) \quad\text{if and only if}\quad r(b_0\varrho_\varkappa^{-1}, \dots, b_{n-1}\varrho_\varkappa^{-1}).$$

Henceforth, whenever we talk about $A^\varkappa$ as being an $\mathbf{X}$-object, it should be understood that we are referring to $\langle A^\varkappa; \{f^1\colon f\in F\}, \{r^1\colon r\in R\}, T_\varkappa\rangle$ which is $\mathbf{X}$-isomorphic to $D(F_{\mathbf{A}}(\varkappa))$ via $\varrho_\varkappa$.

In this way it makes sense to talk about the set of all $\mathbf{X}$-morphisms from an $\mathbf{X}$-object into $A=A^1$, and if, for all $X\in\mathbf{X}$, $\mathbf{X}(X, A)$ with the pointwise induced operations is a subalgebra of A^X, then $E=\mathbf{X}(-, A)\colon \mathbf{X}^{\mathrm{op}}\to\mathbf{A}$ is a well-defined functor. Furthermore $\eta\colon \mathrm{id}_{\mathbf{A}}\to ED$ and $\varepsilon\colon \mathrm{id}_{\mathbf{X}}\to DE$ are natural transformations, where for all $B\in\mathbf{A}$, all $b\in B$, $b\eta_B=\Gamma_b$, where $g\Gamma_b=bg$ for all $g\in D(B)$, and for all $X\in\mathbf{X}$, all $x\in X$, $x\varepsilon_X=\Gamma_x$, where $\lambda\Gamma_x=x\lambda$ for all $\lambda\in E(X)$.

Consequently $\langle D, E, \eta, \varepsilon\rangle$ is a natural candidate for a duality between $\mathbf{A}$ and $\mathbf{X}$; which motivates the following definition.

1.1 DEFINITION. Let D be a contravariant functor from the prevariety $\mathbf{A}$ generated by a compact topological algebra A into a category $\mathbf{X}$ of structured compact spaces, and suppose that $|D(-)|=\mathbf{A}(-, A)$, where $|-|\colon \mathbf{X}\to\mathbf{Comp}$ is the forgetful functor. If, for each $\varkappa>0$, $\varrho_\varkappa$ is an $\mathbf{X}$-isomorphism from $DF_{\mathbf{A}}(\varkappa)$ into $A^\varkappa$, and for all $X\in\mathbf{X}$, $\mathbf{X}(X, A)$ is a subalgebra of A^X, then $\langle D, E, \eta, \varepsilon\rangle$ is called a *pre-duality* (between $\mathbf{A}$ and $\mathbf{X}$), where $E=\mathbf{X}(-, A)$, and η, ε are defined as above.

Using the above terminology, the following is an immediate consequence of [4, Theorem 2.14].

1.2 THEOREM. *Suppose that* $\langle D, E, \eta, \varepsilon\rangle$ *is a pre-duality between* $\mathbf{A}$ *and* $\mathbf{X}$, *where* $\mathbf{A}$ *is generated by a finite discrete (compact topological) algebra* A. *Then* $\langle D, E, \eta, \varepsilon\rangle$ *is a duality if and only if*

(D_0) *for all* $B\in\mathbf{A}$, *all* $X\in\mathbf{X}$, *and every* $h\in\mathbf{A}(B, E(X))$, *the map* $\psi\colon X\to\mathbf{A}(B, A)$, *given by* $b(x\psi)=x(bh)$, *is an* $\mathbf{X}$*-morphism,*

(D_1) *for all* $B, C\in\mathbf{A}$, all *surjections* $f\in\mathbf{A}(B, C)$, *and every* $\lambda\in\mathbf{X}(D(C), A)$, *there exists* $\bar\lambda\in\mathbf{X}(D(B), A)$ *such that* $\lambda=D(f)\bar\lambda$, *and*

(D_2) *for all* $1\leqq n<\omega$, *every element of* $\mathbf{X}(A^n, A)$ *is a polynomial function.*

Essentially, $\langle D, E, \eta, \varepsilon\rangle\colon \mathbf{A}\to\mathbf{X}^{\mathrm{op}}$ is an adjunction if and only if (D_0) holds; (D_2) states precisely that η_B is an isomorphism whenever B is a free algebra in $\mathbf{A}$, and this property extends to arbitrary algebras in $\mathbf{A}$ via (D_1) (see also [4, Corollary 2.16]). Note that since $\mathbf{A}=ISP(\{A\})$, every algebra $B\in\mathbf{A}$ has enough homomorphisms into A to separate points, and hence η_B is one-to-one.

2. The dual categories A and X

Let $A=\langle W_3; \wedge, \vee, *\rangle$ be the algebra obtained by defining a nullary operation $*$ on W_3 by putting $1=*$. By construction, $\wedge$ and $\vee$ are commutative operations, $x\wedge y=x$ if and only if $x\vee y=y$ for all $x, y\in A$, and in A

$$0\wedge * = 0;\quad *\wedge 2 = *;\quad 2\wedge 0 = 2.$$

If $\mathbf{A}$ is the equational class generated by A, then by forgetting the nullary operation, every algebra in $\mathbf{A}$ becomes a member of $\mathbf{W}_3$. Consequently $\mathbf{A}$ is congruence distributive, the subdirectly irreducibles in $\mathbf{A}$ are $\mathbf{2}^*$, $\mathbf{2}_*$, and A, and $\mathbf{A}=ISP(\{A\})$ [9, Corollary 3.4], where $\mathbf{2}^*$ and $\mathbf{2}_*$ denote the two element distributive lattice with, respectively, the unit element and the zero element as the nullary operation $*$.

Now let $B\in\mathbf{W}_3$, and let b be any element of B. By the natural symmetry of W_3 it is clear that there is an embedding φ of B as a subdirect product into $\prod(B_i|i\in I)$ where, for each $i\in I$, the algebra B_i is one of W_3, $\{1, 2\}$, $\{0, 1\}$, and $(b\varphi)(i)=1$. Thus the algebra obtained from B (by distinguishing b as the nullary operation $*$) is in $\mathbf{A}$, since it can be represented as a subdirect product of $\mathbf{2}_*$, $\mathbf{2}^*$ and A. Therefore the algebras in $\mathbf{A}$ and $\mathbf{W}_3$ are the same, modulo the nullary operation, and consequently any representation theory for $\mathbf{A}$ will almost certainly tell us a great deal about $\mathbf{W}_3$.

As with all weakly associative lattices, each algebra $B\in\mathbf{A}$ is completely characterised by its associated structure $\langle B: *, \leqq\rangle$. As it is often easier to work with these structures, we remind the reader that in A we have $0< * <2<0$.

Let $\mathbf{Y}$ be the category of structured compact spaces whose objects are all those quadruplets $\langle X; *, \overset{p}{\leqq}, T\rangle$; where $*$ is a nullary operation, and $\overset{p}{\leqq}$ a pseudo-order on X. Given an object X in $\mathbf{Y}$, a subset U of X is said to be *increasing* if, for all $x, y\in X$, if $x\in U$ and $x\overset{p}{\leqq}y$, then $y\in U$. *Decreasing* subsets of X are defined dually.

2.1 Definition. Let $X\in\mathbf{Y}$, and suppose that (U_0, U_*, U_2) is a partition of X. Then (U_0, U_*, U_2) is called an *$\mathcal{O}$-partition* of X if

(i) $*\in U_*$,
(ii) U_0 is clopen and decreasing, U_2 is clopen and increasing,
(iii) for all $x\in U_0$ and all $y\in U_2$ we have $x\overset{p}{\leqq}y$.

The set of all $\mathcal{O}$-partitions of X is denoted by $\mathcal{O}(X)$.

We can now define the category $\mathbf{X}$.

2.2 Definition. Let $\mathbf{X}$ be the full subcategory of $\mathbf{Y}$ consisting of all those objects $X\in\mathbf{Y}$ which satisfy the following separation conditions.

Let $x, y\in X$ such that $y\overset{p}{\nleqq}x$;

(S_0) if $*\overset{p}{\nleqq}x$, then there exists $(U_0, U_*, U_2)\in\mathcal{O}(X)$ such that $x\in U_0$, $y\in U_*$,

(S_1) if $y\overset{p}{\nleqq}*$, then there exists $(U_0, U_*, U_2)\in\mathcal{O}(X)$ such that $x\in U_*$, $y\in U_2$,

(S_2) if $x\overset{p}{\nleqq}*\overset{p}{\nleqq}y$, then there exists $(U_0, U_*, U_2)\in\mathcal{O}(X)$ such that $y\in U_0$, $x\in U_2$,

(S_3) if $y\overset{p}{\nleqq}*\overset{p}{\nleqq}x$ and either $*\overset{p}{\nleqq}y$ or $x\overset{p}{\nleqq}*$, then there exists $(U_0, U_*, U_2)\in\mathcal{O}(X)$ such that $x\in U_0$, $y\in U_2$.

As the first step in establishing a functor $D\colon \mathbf{A}\to\mathbf{X}^{\mathrm{op}}$, note that A, with the discrete topology, is trivially a compact topological algebra; in all that follows this topology will remain fixed.

For each $B\in\mathbf{A}$, define $D(B)=\langle\mathbf{A}(B, A); *, \overset{p}{\leqq}, T_B\rangle$, where $*$ denotes the constant homomorphism of B onto $\{*\}$, T_B is the topology inherited from the product space A^B, and $\overset{p}{\leqq}$ is given by

$$f\overset{p}{\leqq}g \quad\text{if and only if}\quad bf\leqq bg \quad\text{for all}\quad b\in B.$$

As noted earlier, $\langle\mathbf{A}(B, A); T_B\rangle$ is a compact topological space, one sub-basis of which is

$$\{U_i(b)\colon i\in A, b\in B\},\quad\text{where}\quad U_i(b)=\{f\in\mathbf{A}(B, A)\colon bf=i\}.$$

Note that for each $i\in A$, $b\in B$, $U_i(b)$ is a clopen set. Furthermore, $U_0(b)$ is decreasing, since if $bg=0$ and $f\overset{p}{\leqq}g$, then either $bf=0$ or $bf=2$, and the latter yields the contradiction

$$(b\wedge *)g=0\wedge *=0<*=2\wedge *=(b\wedge *)f.$$

Dually, $U_2(b)$ is increasing, and by definition, if $f\in U_0(b)$ and $g\in U_2(b)$, then $bf==0\nleqq 2=bg$, so $f\overset{p}{\leqq}g$. Therefore, for each $b\in B$, $(U_0(b), U_*(b), U_2(b))$ is an $\mathcal{O}$-partition of $D(B)$, which, coupled with the next result, tells us that $D(B)$ is an object of $\mathbf{X}$ for all $B\in\mathbf{A}$.

2.3 Lemma. *Let $B\in\mathbf{A}$, and let $f, g\in D(B)$ such that $g\overset{p}{\leqq}f$. Then the following conditions hold:*

(0) *if $*\overset{p}{\nleqq}f$, then there exists $b\in B$ with $bf=0$, $bg=*$;*

(1) *if $g\overset{p}{\nleqq}*$, then there exists $b\in B$ with $bf=*$, $bg=2$;*

(2) *if $f\overset{p}{\nleqq}*\overset{p}{\nleqq}g$, there exists $b\in B$ with $bg=0$, $bf=2$;*

(3) *if $g\overset{p}{\nleqq}*\overset{p}{\nleqq}f$ and either $f\overset{p}{\nleqq}*$ or $*\overset{p}{\nleqq}g$, then there exists $b\in B$ with $bf=0$, $bg=2$,*

Proof. Let $f, g\in\mathbf{A}(B, A)$ such that $g\overset{p}{\nleqq}f$. Since (0) and (1) are proved by dual arguments, we prove only (0). Suppose $*\overset{p}{\nleqq}f$. Therefore there exists $c\in B$ with $cf=0$. If $cg=*$ we are finished; so suppose that either $cg=0$ or $cg=2$. But if the latter holds, then $(c\wedge *)f=0$ and $(c\wedge *)g=*$; so we can assume that $(cf, cg)=(0, 0)$. Now $g\overset{p}{\nleqq}f$, so for some $d\in B$, $df<dg$. If $(df, dg)=(*, 2)$, then

$$([d\wedge c]\wedge *)f=0 \quad\text{and}\quad ([d\wedge c]\wedge *)g=*,$$

and if $(df, dg)=(2, 0)$, then

$$([d\vee *]\vee c)f=0 \quad\text{and}\quad ([d\vee *]\vee c)g=*.$$

Since the only other possibility is $(df, dg)=(0, *)$, this completes the proof of (0).

For (2), suppose that $f\overset{p}{\nleqq}*\overset{p}{\nleqq}g\overset{p}{\nleqq}f$. Therefore there exist $a, c, d\in B$ with $af=2$, $cg=0$, and $df<dg$. As before we can suppose that $cf=0$ or $*$; $ag=2$ or $*$; and $(df, dg)\in\{(0,*), (*, 2)\}$.

If $(df, dg)=(0, *)$, then putting $b=(a\wedge d)\wedge(c\wedge d)$ gives

$$bf=(2\wedge 0)\wedge([0 \text{ or } *]\wedge 0)=(2\wedge 0)\wedge 0=2,$$

and

$$bg=([2 \text{ or } *]\wedge *)\wedge(0\wedge *)=*\wedge 0=0.$$

Dually, if $(df, dg)=(*, 2)$, then putting $b=(a\vee d)\vee(c\vee d)$ gives $bf=2$, $bg=0$.

Finally, to prove (3), suppose that $g\overset{p}{\nleqq}*\overset{p}{\nleqq}f$ and that $f\overset{p}{\nleqq}*$. Therefore there exist $x, y\in B$ with $xf=2=yg$. Putting $a=(x\vee *)\vee(y\vee *)$ gives $(af, ag)=(2,2)$. Also there exists $z\in B$ with $zf=0$. Putting $c=z\wedge *$ gives $(cf, cg)=(0, 0 \text{ or } *)$. Now suppose

$d\in B$ such that $df<dg$. If $(df, dg)=(0, *)$, then $(d\vee a)f=0\vee 2=0$, and $(d\vee a)g= =*\vee 2=2$. This tells us that we need bother only about the case $(cf, cg)=(0, 0)$; and if $(df, dg)=(*, 2)$, then a dual argument to the above tells us that $(d\wedge c)f=0$ and $(d\wedge c)g=2$. Finally, let $(df, dg)=(2, 0)$. Again supposing that $(cf, cg)=(0, 0)$, putting $b=c\wedge([d\wedge *]\wedge a)$ gives

$$bf = 0\wedge([2\wedge *]\wedge 2) = 0\wedge * = 0,$$

and

$$bg = 0\wedge([0\wedge *]\wedge 2) = 0\wedge(0\wedge 2) = 2.$$

This completes the proof of the lemma, since the other half of (3) follows by duality.

In accordance with our stated plan, for each $B, C\in\mathbf{A}$ and any $f\in\mathbf{A}(B, C)$, put $D(f)=\mathbf{A}(f, A)$; that is, $gD(f)=fg$ for all $g\in\mathbf{A}(C, A)$. Defined in this way, $D(f)$ is continuous, since for each sub-basic open $U_i(b)$ in $D(B)$, the inverse image $U_i(b)D(f)^{-1}=U_i(bf)$ is a sub-basic open set in $D(C)$. To prove that $D(f)$ is structure-preserving, suppose $g, h\in D(C)$ such that $g\overset{p}{\leqq}h$. Therefore, for all $b\in B$,

$$b\big(gD(f)\big) = b(fg) = (bf)g \leqq (bf)h = b(fh) = b\big(hD(f)\big)$$

by definition of $\overset{p}{\leqq}$ in $D(C)$. Therefore, by definition of $\overset{p}{\leqq}$ in $D(B)$, $gD(f)\overset{p}{\leqq}hD(f)$. Now let $*', *''$ be the nullary operations on $D(B)$, $D(C)$ respectively. Hence, for all $b\in B$, $b\big(*''D(f)\big)=(bf)*''=*$ by definition of $*''$; so $*''D(f)=*'$ is the constant homomorphism of B onto $\{*\}$.

2.4 Corollary. *D: $\mathbf{A}\to\mathbf{X}^{\mathrm{op}}$ is a well-defined functor such that $|D(-)|=\mathbf{A}(-, A)$.*

As mentioned in Section 1, for each $\varkappa>0$, we could now define a structure on $A^\varkappa$, via the homeomorphism $\varrho_\varkappa$, so that it becomes an $\mathbf{X}$-object. Instead, it will be more convenient to start from the other end; that is, we will directly define a structure on $A^\varkappa$ to obtain a $\mathbf{Y}$-object $\langle A^\varkappa; *, \overset{p}{\leqq}, T_\varkappa\rangle$, and then show that $\varrho_\varkappa: D(F_{\mathbf{A}}(\varkappa))\to\langle A^\varkappa; *, \overset{p}{\leqq}, T_\varkappa\rangle$ is a $\mathbf{Y}$-isomorphism. Consequently, since $\varrho_\varkappa$ preserves $\mathcal{O}$-partitions, $A^\varkappa$ becomes an $\mathbf{X}$-object, and $\varrho_\varkappa$ an $\mathbf{X}$-isomorphism. In order to do this we need to know a little about the structure of $\mathbf{X}$-objects.

2.5 Lemma. *Let $X\in\mathbf{X}$, and $x, y\in X$. If $x\neq y$ and $x\overset{p}{\leqq}y$, then either $x, y\overset{p}{\leqq}*$, or $*\overset{p}{\leqq}x, y$, or $x\overset{p}{\leqq}*\overset{p}{\leqq}y$.*

Proof. Suppose $x\neq y$ and $x\overset{p}{\leqq}y$, and that the conclusion does not hold. Therefore $x\neq *\neq y$, $y\overset{p}{\nleqq}x$, and either

(a) $y\overset{p}{\leqq}*\overset{p}{\leqq}x$,

or

(b) $x\overset{p}{\nleqq}*\overset{p}{\nleqq}x$ or $y\overset{p}{\nleqq}*\overset{p}{\nleqq}y$.

If (a), then by (S_2) of Definition 2.2, there exists $(U_0, U_*, U_2)\in\mathcal{O}(X)$ with $x\in U_2$, $y\in U_0$. But U_2 is increasing and $x\overset{p}{\leqq}y$, which is a contradiction.

If (b), then by duality suppose that $x\overset{p}{\nleqq}*\overset{p}{\nleqq}x$. If $y\overset{p}{\nleqq}*$, then by (S_3) of Definition 2.2, there exists $(U_0, U_*, U_2)\in\mathcal{O}(X)$ with $y\in U_2$, $x\in U_0$; but $x\overset{p}{\leqq}y$, which contradicts Definition 2.1. On the other hand, if $y\overset{p}{\leqq}*$, then by hypotheses, $x\overset{p}{\nleqq}*\overset{p}{\nleqq}y$, so as before, (S_2) provides a contradiction.

Hence objects in $\mathbf{X}$ may be visualised as in Figure 1:

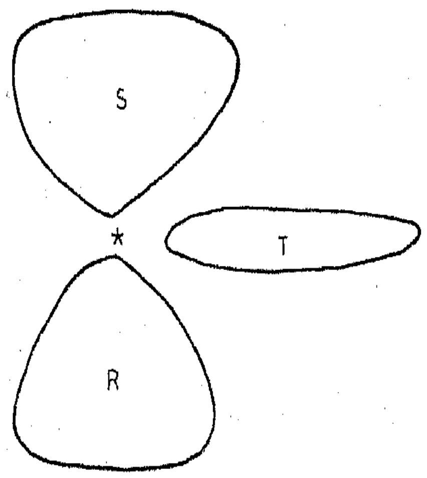

Fig. 1

$$R = \{x\in X:\ x\overset{p}{<}*\},\quad S = \{x\in X:\ *\overset{p}{<}x\},\quad T = \{x\in X:\ x\overset{p}{\nleqq}* \text{ and } *\overset{p}{\nleqq}x\}.$$

It follows from the lemma that the only strict order relations occur within $R\cup\{*\}$, $S\cup\{*\}$, or from R up to S. Consequently T is discretely ordered and the elements of T are not related to the elements of R and S.

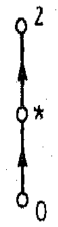

Fig. 2

This yields a clue as to how to define an appropriate pseudo-order on powers of A. Throughout the following, $\leqq$ denotes the pseudo-order on A and its pointwise extension to $A^\varkappa$.

2.6 Definition. Let $\varkappa>0$ and define $\overset{p}{\leqq}$ on $A^\varkappa$ by: for all $a, b\in A^\varkappa$, $a\overset{p}{\leqq}b$ if and only if

$$a=b \text{ or } (a\leqq b \text{ and either } a, b\leqq *, \text{ or } *\leqq a, b, \text{ or } a\leqq *\leqq b).$$

Note, in particular, that as a $\mathbf{Y}$-object, A is pseudo-ordered by $0\overset{p}{<}*\overset{p}{<}2$, as in Figure 2.

Since $\overset{p}{\leqq}$ is a proper subset of $\leqq$, it is clearly a pseudo-order on $A^\varkappa$, so $\langle A^\varkappa; *, \overset{p}{\leqq}, T_\varkappa\rangle$ is a $\mathbf{Y}$-object. To prove that it is also an $\mathbf{X}$-object we need the following pair of results.

2.7 Proposition. *Let $p: A^\varkappa\to A$ be any polynomial function. If $a, b\in A^\varkappa$ such that $a\overset{p}{\leqq}b$, then $p(a)\overset{p}{\leqq}p(b)$.*

Proof. Suppose the result is true for polynomial functions p and q, and let $a, b\in A^\varkappa$ such that $a\overset{p}{\leqq}b$. By duality it is enough to show that $p(a)\vee q(a)\overset{p}{\leqq}p(b)\vee q(b)$. If $a, b\leqq *$ or $*\leqq a, b$, this is clearly true, since all algebraic functions on the two element chain preserve $\leqq$; so suppose that $a \leqq * \leqq b$ and $a \leqq b$. By hypothesis, $p(a)$, $q(a) \leqq *$ and $* \leqq p(b)$, $q(b)$, so clearly $p(a) \vee q(a) \leqq * \leqq p(b) \vee q(b)$. Hence we need show only that $(p(a) \vee q(a), p(b) \vee q(b)) \neq (0,2)$. But $p(a) \vee q(a) = 0$ implies that $p(a) = 0 = q(a)$ (since $p(a)$, $q(a) \leqq *$), and similarly $p(b) \vee q(b) = 2$ implies that $p(b) = 2$ or $q(b) = 2$. But by hypothesis, $p(a) \leqq p(b)$ and $q(a) \leqq q(b)$, so $(0,2) \notin \{(p(a), p(b)), (q(a), q(b))\}$.

2.8 Lemma. *For each* $\varkappa > 0$, *all* $f, g \in \mathbf{A}(F_{\mathbf{A}}(\varkappa), A)$, $f \overset{p}{\leqq} g$ *in* $D(F_{\mathbf{A}}(\varkappa))$ *if and only if* $f\varrho_\varkappa \overset{p}{\leqq} g\varrho_\varkappa$ *in* $\langle A^\varkappa; \overset{p}{\leqq} \rangle$.

Proof. If $f \overset{p}{\leqq} g$ in $D(F_{\mathbf{A}}(\varkappa))$, then for all $x \in F_{\mathbf{A}}(\varkappa)$, $xf \leqq xg$. By Lemma 2.5 and Definition 2.6 this implies that $\langle x_\gamma f \rangle_{\gamma < \varkappa} \overset{p}{\leqq} \langle x_\gamma g \rangle_{\gamma < \varkappa}$, where $\{x_\gamma : \gamma < \varkappa\}$ freely generates $F_{\mathbf{A}}(\varkappa)$. Therefore, by definition of $\varrho_\varkappa$, we have $f\varrho_\varkappa \overset{p}{\leqq} g\varrho_\varkappa$.

Conversely, if $\langle x_\gamma f \rangle_{\gamma < \varkappa} \overset{p}{\leqq} \langle x_\gamma g \rangle_{\gamma < \varkappa}$, then by the previous proposition $p\langle x_\gamma f \rangle_{\gamma < \varkappa} \overset{p}{\leqq} p\langle x_\gamma g \rangle_{\gamma < \varkappa}$, for any polynomial function $p \colon A^\varkappa \to A$. Therefore, since f and g are homomorphisms, $af \overset{p}{\leqq} ag$ for any $a \in F_{\mathbf{A}}(\varkappa)$; hence $af \leqq ag$ for all $a \in F_{\mathbf{A}}(\varkappa)$, so $f \overset{p}{\leqq} g$ in $D(F_{\mathbf{A}}(\varkappa))$.

2.9 Corollary. *For each* $\varkappa > 0$, *the map* $\varrho_\varkappa$ *is an* $\mathbf{X}$*-isomorphism from* $D(F_{\mathbf{A}}(\varkappa))$ *to* $\langle A^\varkappa; *, \overset{p}{\leqq}, T_\varkappa \rangle$.

Henceforth we will denote the $\mathbf{X}$-object $\langle A^\varkappa; *, \overset{p}{\leqq}, T_\varkappa \rangle$ by $A^\varkappa$. Thus $\mathbf{X}(X, A)$ is a well-defined object for each $X \in \mathbf{X}$, and in fact becomes an algebra in $\mathbf{A}$ when $\wedge$, $\vee$ and $*$ are defined pointwise. Clearly $*$, the constant map of X onto $\{*\}$, is an element of $\mathbf{X}(X, A)$. Now suppose that $\gamma, \sigma \in \mathbf{X}(X, A)$; by duality it will suffice to show that $\gamma \wedge \sigma \in \mathbf{X}(X, A)$. Firstly, the continuity of $\gamma \wedge \sigma$ follows from the fact that A is a topological algebra; for if $x \in X$, and U is any neighbourhood of $x(\gamma \wedge \sigma) = x\gamma \wedge x\sigma$, then by the continuity of $\wedge$, there exist neighbourhoods V_1, V_2 of $x\gamma$, $x\sigma$ respectively such that for all $a, b \in A$, $(a, b) \in V_1 \times V_2$ implies that $a \wedge b \in U$. But γ, σ are continuous on X, so there exist neighbourhoods W_1 and W_2 of x in X such that for all $y, z \in X$,

$$(y, z) \in W_1 \times W_2 \text{ implies that } (y\gamma, z\sigma) \in V_1 \times V_2.$$

Hence, for all $y \in X$, if $y \in W_1 \cap W_2$, then $y(\gamma \wedge \sigma) = y\gamma \wedge y\sigma \in U$.

Clearly $\gamma \wedge \sigma$ preserves $*$, since $*(\gamma \wedge \sigma) = *\gamma \wedge *\sigma = * \wedge * = *$. Finally, to see that $\gamma \wedge \sigma$ preserves $\overset{p}{\leqq}$, suppose $x, y \in X$ such that $x \overset{p}{\leqq} y$ and $x \neq y$. By Lemma 2.5, either $x, y \overset{p}{\leqq} *$, or $* \overset{p}{\leqq} x, y$ or $x \overset{p}{\leqq} * \overset{p}{\leqq} y$. Since all cases are dealt with similarly, we can suppose that $x, y \overset{p}{\leqq} *$. Therefore $(x\gamma, x\sigma) \leqq *$ and $(y\gamma, y\sigma) \leqq *$ in A^2, and $x\gamma \leqq y\gamma$, $x\sigma \leqq y\sigma$ in A. Hence $(x\gamma, x\sigma) \leqq (y\gamma, y\sigma)$ in A_2, so by Proposition 2.7,

$$x(\gamma \wedge \sigma) = x\gamma \wedge x\sigma \overset{p}{\leqq} y\gamma \wedge y\sigma = y(\gamma \wedge \sigma)$$

as required.

It is now routine to verify that for all $X, Y \in \mathbf{X}$, and every $\gamma \in \mathbf{X}(X, Y)$, the map $\mathbf{X}(\gamma, A) \colon \mathbf{X}(Y, A) \to \mathbf{X}(X, A)$ is a homomorphism, and consequently Corollaries 2.4 and 2.9 yield our next result.

2.10 Lemma. $E = \mathbf{X}(-, A) \colon \mathbf{X}^{\mathrm{op}} \to \mathbf{A}$ *is a well-defined functor, and* $\langle D, E, \eta, \varepsilon \rangle$ *is a pre-duality between* $\mathbf{A}$ *and* $\mathbf{X}$.

The main result of this paper can now be stated.

2.11 Theorem. $\langle D, E, \eta, \varepsilon \rangle$ *is a full duality between* $\mathbf{A}$ *and* $\mathbf{X}$.

3. Proof of Theorem 2.11

Firstly, in order to show that $\langle D, E, \eta, \varepsilon \rangle$ is a duality, we need to verify that conditions (D_0), (D_1) and (D_2) of Theorem 1.2 hold. For (D_0), suppose that $B \in \mathbf{A}$, $X \in \mathbf{X}$, $h \in \mathbf{A}(B, E(X))$, and that $\psi \colon X \to \mathbf{A}(B, A)$ is given by

$$b(x\psi) = x(bh) \quad \text{for all} \quad x \in X, \quad b \in B.$$

Observe that for any sub-basic open set $U_i(b)$ in $\mathbf{A}(B, A)$ with $b \in B$, $i \in A$, we have

$$U_i(b)\psi^{-1} = \{x \in X \colon b(x\psi) = i\} = \{x \in X \colon x(bh) = i\}$$

is open in X since bh is continuous from X into A. Now suppose that $x, y \in X$ such that $x \overset{p}{\leqq} y$. Therefore, $b(x\psi) = x(bh) \overset{p}{\leqq} y(bh) = b(y\psi)$ for all $b \in B$ since bh preserves $\overset{p}{\leqq}$. Similarly, $b(*\psi) = *(bh) = *$ for all $b \in B$ since bh preserves $*$, and consequently $\psi \in \mathbf{X}(X, D(B))$ as required.

For the verification of (D_1) and (D_2), we need to know that for each $X \in \mathbf{X}$, there is a natural bijection between $\mathcal{O}(X)$ and $\mathbf{X}(X, A)$.

3.1 Proposition. *Let* $X \in \mathbf{X}$. *Each element* (U_0, U_*, U_2) *of* $\mathcal{O}(X)$ *gives rise to an element* γ *of* $\mathbf{X}(X, A)$, *where for all* $x \in X$, $x\gamma = i$ *if and only if* $x \in U_i$. *Conversely, if* $\gamma \in \mathbf{X}(X, A)$, *then* $(U_0, U_*, U_2) \in \mathcal{O}(X)$, *where* $x \in U_i$ *if and only if* $x\gamma = i$.

Proof. Suppose $(U_0, U_*, U_2) \in \mathcal{O}(X)$, and γ is defined as indicated. Since each U_i is clopen, and $* \in U_*$, γ is continuous and $*$-preserving. Suppose $x, y \in X$ such that $x \overset{p}{\leqq} y$. If $x \in U_2$ or $y \in U_0$, then since U_2 is increasing, and U_0 is decreasing, we must have $x\gamma = y\gamma$. Otherwise, $x \in U_* \cup U_0$ and $y \in U_2 \cup U_*$. But by Definition 2.1 (iii), $(x, y) \notin U_0 \times U_2$, and hence either $x, y \in U_*$, in which case $x\gamma = y\gamma$, or $(x, y) \in (U_0 \times U_*) \cup \cup (U_* \times U_2)$. Therefore, in all cases, $x\gamma \overset{p}{\leqq} y\gamma$.

Conversely, if $\gamma \in \mathbf{X}(X, A)$, and (U_0, U_*, U_2) is defined as indicated, then since γ is continuous, each U_i is clopen in X. Clearly $* \in U_*$, and for all $(x, y) \in U_0 \times U_2$, $x \overset{p}{\nleqq} y$. By duality, it remains to show that U_0 is decreasing, so suppose $x \overset{p}{\leqq} y$, where $y\gamma = 0$. Therefore $x\gamma \overset{p}{\leqq} y\gamma = 0$, so by definition of $\overset{p}{\leqq}$ in A, $x\gamma = 0$ as required.

3.2 Lemma. *For each* $1 \leqq n < \omega$, *every element of* $\mathbf{X}(A^n, A)$ *is a polynomial function.*

Proof. Let $\varphi \in \mathbf{X}(A^n, A)$. Since the polynomial $m(x, y, z) = ([x \wedge y] \vee [x \wedge z]) \vee \vee (y \wedge z)$ satisfies the 'near unanimity' identities in any weakly associative lattice [7], by [1, p. 168] it suffices to show that for all $a = \langle a_0, \ldots, a_{n-1} \rangle$, $b = \langle b_0, \ldots, b_{n-1} \rangle \in A^n$, there exists a polynomial function $p \colon A^n \to A$ such that $p(a_0, \ldots, a_{n-1}) = a\varphi$ and $p(b_0, \ldots, b_{n-1}) = b\varphi$. By duality and the fact that $*$ is a nullary operation, it suffices to deal with the cases $(a\varphi, b\varphi) \in \{(0, 2), (0, 0), (0, *)\}$.

Case (i). Suppose $(a\varphi, b\varphi) = (0, 2)$. By Corollary 2.9, $\varrho_n\varphi \in \mathbf{X}(D(F_{\mathbf{A}}(n)), A)$, so by the previous proposition, (U_0, U_*, U_2) is an element of $\mathcal{O}(X)$, where U_i is the preimage of $\{i\}$ under $\varrho_n\varphi$ for each $i \in A$. Now $(a\varrho_n^{-1}, b\varrho_n^{-1}) \in U_0 \times U_2$, and hence, by Definition 2.1, we have $b\varrho_n^{-1} \overset{p}{\nleqq} * \overset{p}{\nleqq} a\varrho_n^{-1} \overset{p}{\nleqq} b\varrho_n^{-1}$ in $D(F_{\mathbf{A}}(n))$. Therefore, by Lemma 2.3 (2), there exists $x \in F_{\mathbf{A}}(n)$ such that $x(b\varrho_n^{-1}) = 2$, $x(a\varrho_n^{-1}) = 0$. But $x = p(x_0, \ldots, x_{n-1})$ for some polynomial p, where $x_0, \ldots, x_{n-1}$ are the free generators of $F_{\mathbf{A}}(n)$.

Hence, $p(x_0(b\varrho_n^{-1}), \ldots, x_{n-1}(b\varrho_n^{-1})) = 2$ and $p(x_0(a\varrho_n^{-1}), \ldots, x_{n-1}(a\varrho_n^{-1}))_p = 0$, since $a\varrho_n^{-1}, b\varrho_n^{-1} \in \mathbf{A}(F_{\mathbf{A}}(\varkappa), A)$. That is, by definition of ϱ_n, $p(a_0, \ldots, a_{n-1}) = 0$ and

$p(b_0, \ldots, b_{n-1})=2$. The case $(a\varphi, b\varphi)=(0, *)$ is similarly dealt with via Lemma 2.3 (0).

Case (ii). Suppose $(a\varphi, b\varphi)=(0, 0)$. Hence $* \overset{p}{\nleqq} a, b$; so there exist j, k such that $a_j=0=b_k$. Therefore $(a_j\wedge *)\wedge(a_k\wedge *)=0=(b_j\wedge *)\wedge(b_k\wedge *)$, which completes the proof.

Finally, in order to check that (D_1) holds, let $B, C\in\mathbf{A}$ and suppose that $f\in\mathbf{A}(B, C)$ is a surjection. By [4, Proposition 2.6], $D(f)\in\mathbf{X}(D(C), D(B))$ is a homeomorphism onto a closed subspace, and if $g, h\in D(C)$ such that $gD(f)\overset{p}{\leqq} hD(f)$, then for all $b\in B$,

$$(bf)g = b(fg) = b(gD(f)) \leqq b(hD(f)) = b(fh) = (bf)h;$$

since f is onto it follows that $g\overset{p}{\leqq} h$ in $D(C)$. Thus $D(f)$ is an $\mathbf{X}$-isomorphism of $D(C)$ onto a closed subspace of $D(B)$; so clearly, in order to verify that for all $\gamma\in\mathbf{X}(D(C), A)$ there exists $\bar{\gamma}\in\mathbf{X}(D(B), A)$ such that $\gamma=D(f)\bar{\gamma}$, it suffices to show that A is *injective in* $\mathbf{X}$ in the sense of the following result.

3.3 Lemma. *Let $X, Y\in\mathbf{X}$ such that X is a closed substructure of Y. For each $\gamma\in\mathbf{X}(X, A)$, there exists $\bar{\gamma}\in\mathbf{X}(Y, A)$ such that $\bar{\gamma}\upharpoonright X=\gamma$.*

Proof. Let $\gamma\in\mathbf{X}(X, A)$, and let (W_0, W_*, W_2) be the $\mathcal{O}$-partition of X determined by putting $W_i=\{x\in X\colon x\gamma=i\}$. Let $y\in W_2$, and $x\in W_0\cup W_*$. If $x\in W_*$, then $y \overset{p}{\nleqq} x$ and $y \overset{p}{\nleqq} *$, so by (S_1) of Definition 2.2, there exists $(W_0^{xy}, W_*^{xy}, W_2^{xy})\in\mathcal{O}(Y)$ such that $x\in W_*^{xy}$, $y\in W_2^{xy}$. Similarly, if $x\in W_0$, then $x\overset{p}{\nleqq} y$ and $y\overset{p}{\nleqq} * \overset{p}{\nleqq} x$, so by (S_2) of Definition 2.2 there exists $(W_0^{xy}, W_*^{xy}, W_2^{xy})\in\mathcal{O}(Y)$ such that $x\in W_0^{xy}$ and $y\in W_2^{xy}$. Therefore, by the compactness of W_0 and W_* in Y, for each $y\in W_2$ there exist $x_1, \ldots, x_{n_y}\in W_0\cup W_*$ such that

$$W_0 \subseteqq \bigcup_{i=1}^{n_y} W_0^{x_i y} \quad\text{and}\quad W_* \subseteqq \bigcup_{i=1}^{n_y} W_*^{x_i y}.$$

Put $W_2^y=\bigcap_{i=1}^{n_y} W_2^{x_i y}$ for each $y\in W_2$. Therefore, by the compactness of W_2 in Y, there exist $y_1, \ldots, y_n\in W_2$ such that $W_2\subseteqq\bigcup_{i=1}^{n} W_2^{y_i}=U_2$.

Now let $z\in U_2$, $x\in W_0$. Therefore $z\in W_2^y$ for some $y\in\{y_1, \ldots, y_n\}$, and since $W_0\subseteqq\bigcup_{i=1}^{n_y} W_0^{x_i y}$, we have $x\in W_0^{x_i y}$ for some $i\in\{1, \ldots, n_y\}$. But by definition of W_2^y, also $z\in W_2^{x_i y}$, and hence $x\overset{p}{\nleqq} z$. Since finite unions of finite intersections of clopen increasing subsets of Y are clopen increasing, (W_0, W_*, U_2) is an $\mathcal{O}$-partition of the closed subspace $X^1=(W_0\cup W_*\cup U_2)$ of Y such that $U_2\cap X=W_2$. Dually there is a clopen decreasing subset U_0 of Y such that $U_0\cap X=U_0\cap X^1=W_0$, and for all $x\in U_0$, $y\in U_2$, $x\overset{p}{\nleqq} y$. That is, if $U_*=Y\setminus(U_0\cup U_2)$, then (U_0, U_*, U_2) is an element of $\mathcal{O}(Y)$ such that for all $i\in A$, $U_i\cap X=W_i$. Proposition 3.1 completes the proof.

To summarise the results of the section so far, we have

3.4 Lemma. $\langle D, E, \eta, \varepsilon\rangle\colon \mathbf{A}\to\mathbf{X}^{\mathrm{op}}$ *is a duality between* $\mathbf{A}$ *and* $\mathbf{X}$.

The following lemma completes the proof of Theorem 2.11.

3.5 Lemma. *For each $X\in\mathbf{X}$, $\varepsilon_X\in\mathbf{X}(X, DE(X))$ given by $x\varepsilon_X=\Gamma_x$, where $\gamma\Gamma_x=x\gamma$, is an $\mathbf{X}$-isomorphism.*

Proof. Let $X\in\mathbf{X}$. Clearly, to prove that ε_X is one-to-one, it suffices to show that X has enough $\mathbf{X}$-morphisms into A to separate points. Let $x, y\in X$ such that $x\neq y$. Since $\overset{p}{\leqq}$ is reflexive, either $x\overset{p}{\nleqq} y$ or $y\overset{p}{\nleqq} x$. Without loss of generality suppose that $y\overset{p}{\nleqq} x$. If $y\overset{p}{\nleqq} *$ or $*\overset{p}{\nleqq} x$, then Proposition 3.1 and conditions (S_1), (S_0) respectively of Definition 2.2 provide a $\gamma\in\mathbf{X}(X, A)=E(X)$ such that $x\gamma<y\gamma$. Otherwise $y\overset{p}{\leqq} * \overset{p}{\leqq} x$, and since $y\overset{p}{\nleqq} x$, both $y\neq *$ and $x\neq *$, whence $x\overset{p}{\nleqq} * \overset{p}{\nleqq} y$; consequently, in a similar fashion, (S_2) provides a $\gamma\in\mathbf{X}(X, A)$ such that $x\gamma<y\gamma$. Thus, ε_X is one-to-one; in fact it is an $\mathbf{X}$-isomorphism onto a closed subspace of $DE(X)$, for if $y, x\in X$ such that $y\overset{p}{\nleqq} x$, by what we have just seen there is an element $\gamma\in\mathbf{X}(X, A)$ such that $x\gamma<y\gamma$. That is, $y\gamma\overset{p}{\nleqq} y\gamma$, and so $\gamma\Gamma_y\overset{p}{\nleqq}\gamma\Gamma_x$. Therefore, $\Gamma_y\overset{p}{\nleqq}\Gamma_x$ by the definition of $\overset{p}{\leqq}$ in $DE(X)$. (Note that ε_X is a homeomorphism onto a closed subspace since all objects in $\mathbf{X}$ are compact Hausdorff spaces.)

To complete the proof it suffices to show that if B is any subalgebra of $E(X)=$ $=\mathbf{X}(X, A)$, then every element of $\mathbf{A}(B, A)$ is the restriction of a Γ_x for some $x\in X$. To this end, let B be a subalgebra of $E(X)$ and let $f\in\mathbf{A}(B, A)$; if f is the constant homomorphism onto $\{*\}$, then clearly $f=\Gamma_*\upharpoonright B$. Therefore we may assume that $f\neq *$. This tells us that $B/\ker f$ is subdirectly irreducible, and so, using Jónsson's Lemma and the argument in [11, Proposition 3.2], there exists a unique element $x\in X$ such that for all $\gamma, \sigma\in B$, if $x\gamma=x\sigma$ then $\gamma f=\sigma f$. We will show that $\gamma f=x\gamma$ for all $\gamma\in B$, whence $f=\Gamma_x\upharpoonright B$.

Let $\gamma f=2$. Then $x\gamma\neq 2$ implies that $x(\gamma\vee *)=*=x*$, which yields the contradiction $2=\gamma f=\gamma f\vee *=(\gamma\vee *)f=*f=*$. Dually, $\gamma f=0$ implies that $x\gamma=0$ for all $\gamma\in B$.

Lastly, suppose that $\gamma f=*$. Since $f\neq *$, there exists $\lambda\in B$ with $\lambda f\in\{0, 2\}$; without loss of generality suppose that $\lambda f=0$. Therefore $x\lambda=0$, and since $x\gamma\neq *$ implies that $x(\gamma\vee\lambda)=x\lambda$, which yields the contradiction $*=\gamma f=(\gamma\vee\lambda)f=\lambda f=0$, it follows that $x\gamma=*$. Hence $\gamma f=x\gamma$, and the proof of Theorem 2.11 is complete.

4. Applications

Structure of finite algebras in A and W_3. In order to describe the finite algebras in $\mathbf{A}$ we need the following result, which relies upon the fact that W_3 is injective in $\mathbf{W}_3$ (see [3], [7], or [12]).

4.1 Proposition. *Let $B, C\in\mathbf{A}$, and $f\in\mathbf{A}(B, C)$.*

(i) f is an injection if and only if $D(f)$ is a surjection.

(ii) f is a surjection if and only if $D(f)$ is an $\mathbf{X}$-isomorphism from $D(C)$ onto a closed subspace of $D(B)$.

Proof. (i) If f is an injection and $g\in D(B)=\mathbf{A}(B, A)$, then since A is injective in $\mathbf{A}$, there exists $h\in D(C)$ such that $fh=g$; that is such that $hD(f)=g$. The converse follows since surjections in $\mathbf{X}$ are epic and, since it is a contravariant hom-functor, E sends epics to injections.

(ii) Since $\langle D, E, \eta, \varepsilon\rangle$ is a duality it is sufficient to prove that E maps $\mathbf{X}$-embeddings to surjections. Since, by Lemma 3.3 A is injective in $\mathbf{X}$, this follows as in (i).

Let $\mathbf{V}$ be an equational class, $\mathbf{K}\subseteqq\mathbf{V}$, and $B\in\mathbf{V}$. An algebra $M\in\mathbf{K}$ is a *maximal homomorphic image* of B in $\mathbf{K}$ if there exists a surjective homomorphism $\lambda\colon B\to M$ such that for all $N\in\mathbf{K}$ and surjective homomorphisms $f\colon B\to N$, there exists a unique

$\bar{f}\in\mathbf{V}(M,N)$ such that $f=\lambda\bar{f}$. Let $\mathbf{D}_*=HSP(\{2^*,2_*\})$ be the equational class of all pointed distributive lattices.

4.2 THEOREM. *Let B be a finite algebra in* $\mathbf{A}$. *Then $B\cong M\times A^k$, where M is a maximal homomorphic image in $\mathbf{D}_*$ of B, and $k\geqq 0$.*

PROOF. Consider the following two sets:

$$X_1=\{f\in D(B):*\overset{p}{\leqq}f \text{ or } f\overset{p}{\leqq}*\},\quad X_2=\{f\in D(B):*\overset{p}{\nleq}f \text{ and } f\overset{p}{\nleq}*\}.$$

Since $D(B)$ is finite and therefore discrete, it follows that both X_1 and X_2 are in $\mathbf{X}$. Furthermore, any pair of $\overset{p}{\leqq}$ and $*$ preserving maps from X_1 and X_2 into A can obviously be patched together to yield an element of $ED(B)$, whence, by Theorem 2.11, $B\cong E(X_1)\times E(X_2)$. But, by Lemma 2.5, no two elements of X_2 are comparable under $\overset{p}{\leqq}$, so $E(X_2)\cong A^{n-1}$, where $|X_2|=n$. That $E(X_1)\in\mathbf{D}_*$ follows since the only subdirectly irreducibles in $\mathbf{D}_*$ are $\mathbf{2}^*$ and $\mathbf{2}_*$. To see that $E(X_1)$ is a maximal homomorphic image of B in $\mathbf{D}_*$, let inc: $X_1\to D(B)$ be the natural inclusion map, and put $\lambda=\eta_B E(\mathrm{inc})$. Then $\lambda\in\mathbf{A}(B,E(X_1))$; by Theorem 2.11 and the proposition above, E(inc) is a surjection, and hence so is λ: $B\to E(X_1)$. Now suppose that $f\in\mathbf{A}(B,N)$ is a surjection, where $N\in\mathbf{D}_*$. Since $N\in\mathbf{D}_*$, for all $g\in\mathbf{A}(N,A)$, either $*\overset{p}{\leqq}g$ or $g\overset{p}{\leqq}*$; hence, by the proposition above, $D(f)$ is an $\mathbf{X}$-isomorphism of $D(N)$ onto a closed subspace of X_1 and $ED(f)$: $E(X_1)\to ED(N)$. Put $\bar{f}=ED(f)\eta_N^{-1}$. Therefore $\bar{f}\in\mathbf{A}(E(X_1),N)$ and $\lambda\bar{f}\in\mathbf{A}(B,N)$; we claim that $\lambda\bar{f}=f$. To see this, let $b\in B$, $g\in D(N)$. By the remark above, $fg=gD(f)\in X_1$, and

$$g\big((b\lambda\bar{f})\eta_N\big)=g\big((\Gamma_b E(\mathrm{inc}))ED(f)\big)=g\big((\mathrm{inc}\ \Gamma_b)ED(f)\big)=$$
$$=g\big(D(f)(\mathrm{inc}\ \Gamma_b)\big)=\big(gD(f)\ \mathrm{inc}\ \Gamma_b=(fg)\Gamma_b=b(fg)=(bf)g=g\Gamma_{bf}.$$

Since g was arbitrary, $(b\lambda\bar{f})\eta_N=\Gamma_{bf}$ for all $b\in B$; that is, $b\lambda\bar{f}=\Gamma_{bf}\eta_N^{-1}=bf$ for all $b\in B$. Thus $\lambda\bar{f}=f$, as required. The uniqueness of $\bar{f}$ follows since λ is onto.

4.3 THEOREM. *For each $n\geqq 0$, $F_{\mathbf{A}}(n)\cong F_{\mathbf{D}^*}(n)\times A^{k(n)}$, where $k(n)=3^n-2^{n+1}+1$.*

PROOF. By Theorem 4.2, $F_{\mathbf{A}}(n)\cong M\times A^{k(n)}$, where $k(n)\geqq 0$ and M is a maximal homomorphic image in $\mathbf{D}_*$ of $F_{\mathbf{A}}(n)$. Let $\{x_1,\dots,x_n\}$ and $\{y_1,\dots,y_n\}$ freely generate $F_{\mathbf{A}}(n)$ and $F_{\mathbf{D}_*}(n)$ in $\mathbf{A}$ and $\mathbf{D}_*$ respectively. It is easily seen that $F_{\mathbf{D}_*}(n)$ is a maximal homomorphic image in $\mathbf{D}_*$ of $F_{\mathbf{A}}(n)$ via the homomorphism induced by the mapping $x_i\to y_i$, and since maximal homomorphic images are unique (up to isomorphism), $F_{\mathbf{D}_*}(n)\cong M$. By the proof of the previous theorem, $k(n)$ is the number of elements of $D\big(F_{\mathbf{A}}(n)\big)$ which are non-comparable with $*$ (under $\overset{p}{\leqq}$). But by Lemma 2.8, $D\big(F_{\mathbf{A}}(n)\big)$ is $\mathbf{X}$-isomorphic to A^n, whence

$$k(n)=|\{\langle a_1,\dots,a_n\rangle\in A^n:\text{ for some } i,j\in\{1,\dots,n\},\ a_i=0 \text{ and } a_j=2\}|.$$

Therefore $k(n)=3^n-|\{a\in A^n:\ *\overset{p}{\leqq}a \text{ or } a\overset{p}{\leqq}*\}|+1=3^n-2^{n+1}+1$.

In fact it is easy to see what the sets of free generators of $F_{\mathbf{D}_*}(n)\times A^{k(n)}$ must look like. They are of the form $\{\langle y_i,a_{i1},\dots,a_{ik(n)}\rangle: i=1,\dots,n\}$, where (a_{ij}) is any $n\times k(n)$ matrix over A such that

$$\{a\in A^n:*\overset{p}{\nleqq}a \text{ and } a\overset{p}{\nleqq}*\}=\{\langle a_{1j},\dots,a_{nj}\rangle: 1\leqq j\leqq k(n)\}.$$

One can prove Theorem 4.3 directly from Lemma 3.2 using this method, and then

obtain Theorem 4.2 as a corollary by observing that since $\mathbf{A}$ is congruence distributive it has factorizable congruences. We opted for the above approach because it provides a better illustration of the nature of the duality between $\mathbf{A}$ and $\mathbf{X}$.

It is now easy to obtain the corresponding results for $\mathbf{W}_3$, i.e. Theorems 4 and 5 of [7]; one simply observes that, as noted earlier, for any $B\in\mathbf{W}_3$, the algebra obtained by distinguishing any element of B as a nullary operation $*$ is in $\mathbf{A}$, and if $|-|$: $\mathbf{A}\to\mathbf{W}_3$ is the forgetful functor, then for all $n\geqq 0$, $|F_{\mathbf{A}}(n)|\cong F_{\mathbf{W}_3}(n+1)$ via any bijection from $\{x_1,\dots,x_n,*\}$ to $\{y_1,\dots,y_{n+1}\}$ (where the x_i's and y_j's freely generate $F_{\mathbf{A}}(n)$ and $F_{\mathbf{W}_3}(n+1)$ in $\mathbf{A}$ and $\mathbf{W}_3$ respectively), and similarly $|F_{\mathbf{D}_*}(n)|\cong F_{\mathbf{D}}(n+1)$.

Congruences in $\mathbf{A}$ and $\mathbf{W}_3$. The lattice of congruences on a bounded distributive lattice is dually isomorphic to the lattice of closed subspaces of its Priestley dual; an analogous result holds here.

4.4 THEOREM. *For any $B\in\mathbf{A}$, the lattice $\mathscr{C}(B)$ of congruences on B is dually isomorphic to the lattice of all closed subspaces of $D(B)$ which contain $*$.*

PROOF. For each $Y\in\mathbf{X}$ such that $Y\subseteqq D(B)$, define a relation Θ_Y on B by

$$\langle a,b\rangle\in\Theta_Y \text{ if and only if } af=bf \text{ for all } f\in Y.$$

Clearly Θ_Y is a congruence on B, for if inc: $Y\to D(B)$ is the natural map, then $\Theta_Y=\ker\big(\eta_B E(\mathrm{inc})\big)$.

On the other hand, for each $\Phi\in C(B)$, let Y_Φ be the subspace of $D(B)$ given by

$$Y_\Phi=\{f\in D(B):\text{ for all } a,b\in B,\ a\equiv b(\Phi)\ \text{ implies }\ af=bf\}.$$

Since $Y_\Phi=D(B/\Phi)D(p)$, where p: $B\to B/\Phi$ is the canonical homomorphism, by Proposition 4.1, Y_Φ is a closed substructure of $D(B)$ and $Y_\Phi\in\mathbf{X}$. Let $\Psi=\Theta_{Y_\Phi}$. Clearly $\Phi\subseteqq\Psi$, and if $a\equiv b(\Psi)$, then by definition $ahD(p)=bhD(p)$ for all $h\in D(B/\Phi)$, so $(ap)h=(bp)h$ for all $h\in D(B/\Phi)$. But every algebra in $\mathbf{A}$ has enough homomorphisms into A to separate points, so for all $a,b\in B$, if $a\equiv b(\Psi)$, then $ap=bp$, that is $a\equiv b(\Phi)$. Hence $\Phi=\Psi$. To complete the proof we must show that for all subspaces X and Y of $D(B)$ with $X,Y\in\mathbf{X}$,

$$X\subseteqq Y\quad\text{if and only if}\quad\Theta_Y\subseteqq\Theta_X.$$

That $\Theta_Y\subseteqq\Theta_X$ whenever $X\subseteqq Y$ is trivial, and for the converse, let us suppose that $X\nsubseteqq Y$. Therefore there exists $f\in X\setminus Y$. Suppose $*\overset{p}{\nleqq}f\overset{p}{\nleqq}*$. Then for all $h\in Y$, there exists $(U_0^h,U_*^h,U_2^h)\in\mathcal{O}(D(B))$ such that $h\in U_*^h$, $f\in U_2^h$, so by the compactness of Y, there exist $h_1,\dots,h_n\in Y$ such that $Y\subseteqq\bigcup_{i=1}^{n}U_*^{h_i}$. Put $U_0=\bigcap_{i=1}^{n}U_0^{h_i}$, $U_2=\bigcap_{i=1}^{n}U_2^{h_i}$, and $U_*=$ $=D(B)\setminus(U_0\cup U_2)$. Then $(U_0,U_*,U_2)\in\mathcal{O}\big(D(B)\big)$ such that $Y\subseteqq U_*$, so if $b\in B$ such that Γ_b corresponds to (U_0,U_*,U_2) via Proposition 3.1 and Theorem 2.11, then for all $g\in Y$, $ag=g\Gamma_a=*=*g$, so $a\equiv *(\Theta_Y)$, while $af=f\Gamma_a=2\neq *f$, so $a\not\equiv *(\Theta_X)$. Otherwise f is comparable with $*$, so without loss of generality $*\overset{p}{\leqq}f$. The proof now proceeds as in [5, Theorem 3.1]. Let $W=\{g\in D(B):\ g\overset{p}{\leqq}f\}\cap Y$; then W is closed in $D(B)$, and if $h\in W$ then $f\overset{p}{\nleqq}h$, so there exists $(U_0^h,U_*^h,U_2^h)\in\mathcal{O}\big(D(B)\big)$ such that $h\in U_*^h$, $f\in U_2^h$. Therefore, by the compactness of W, there exist $h_1,\dots,h_n\in W$ such that $W\subseteqq$ $\subseteqq\bigcup_{i=1}^{n}U_*^{h_i}$. Put $U_2=\bigcap_{i=1}^{n}U_2^{h_i}$ and $U_*=\bigcup_{i=1}^{n}(U_*^h\cup U_0^h)$. Therefore $(\varnothing,U_*,U_2)\in\mathcal{O}\big(D(B)\big)$, so there exists $b\in B$ such that $\Gamma_b\in ED(B)$ determines $(\varnothing,U_*,U_2)$.

Now let $Y'=Y\cup\{f\}$, $U'_*=U_*\cup\{f\}$. Clearly Y' is closed in $D(B)$, and $U'_*\cap Y'$ is closed in Y'. Furthermore,

$$Y'\setminus(U'_*\cap Y')=Y\setminus(U_*\cap Y)=Y'\cap(U_2\cap Y),$$

so in fact $U'_*\cap Y'$ is clopen in Y'. Also, if $h\in D(B)$ such that $h\overset{p}{\leqq}f$, then $h\in U'_*$ since $W\subseteqq U_*$, and if $h, g\in D(B)$ such that $h\overset{p}{\leqq}g\in U_*$, then since U_* is decreasing in $D(B)$, $h\in U'_*$. Hence if $\mu\colon Y'\to A$ is given by

$$x\mu=\begin{cases}2 & \text{if } x\notin U'_*\\ * & \text{if } x\in U'_*\end{cases}$$

then $\mu\in\mathbf{X}(Y', A)$. By Theorem 2.11 and Lemma 3.3, there exists $a\in B$ such that $\Gamma_a\restriction Y'=\mu$. Therefore, for all $g\in Y$, since $g\neq f$, $ag=g\Gamma_a=g\mu=g\Gamma_b=bg$ (since μ agrees with Γ_b on Y), so $a\equiv b(\Theta_Y)$. However, $af=f\Gamma_a=*\neq 2=f\Gamma_b=bf$, so $a\not\equiv b(\Theta_X)$, and hence $\Theta_Y\subsetneqq\Theta_X$.

In case B is a finite algebra in **A**, then $D(B)$ is discrete, and so, by the theorem, $\mathscr{C}(B)$ is dually isomorphic to the lattice of all subsets of $D(B)\setminus\{*\}$.

45. COROLLARY [7; 12, Theorem 1]. *If $B\in\mathbf{A}$ is finite, then $\mathscr{C}(B)\cong\mathbf{2}^n$, where n is the number of non-trivial homomorphisms of B into A.*

Theorem 4.4 also provides us with a new proof of the following [3, 7].

4.6 COROLLARY. **A** *satisfies the congruence extension property.*

PROOF. Let $B, C\in\mathbf{A}$, and suppose that $f\colon B\to C$ is an embedding. Then for any congruence Φ on B, putting $X=Y_\Phi D(f)^{-1}$ gives $a\equiv b(\Phi)$ if and only if $af\equiv bf(\Theta_X)$ for all $a, b\in B$.

In view of the fact that, as we have just seen, many of the important known facts about $\mathbf{W}_3$ admit non-algebraic proofs via this interesting topological duality, one wonders whether the same could be true of all $\mathbf{W}_n$. Certainly one cannot obtain a representation theory for $\mathbf{W}_n (n\geqq 4)$ without substantially modifying the technique we have used here. For instance, if we add just one nullary operation $*$ to W_4, and call the resultant algebra and its equational class B, **B** respectively, then no matter how $*$ is placed, $\mathbf{B}(A, B)$ and $\mathbf{B}(B, B)$, ordered pointwise, will be isomorphic as structured compact spaces, so no duality is possible via this method. On the other hand we cannot add any more than one nullary operation to W_4 without loosing a great deal of information about $\mathbf{W}_4$, since the resultant equational class would contain, at best, only two subdirectly irreducibles. Independently of the authors, H. WERNER [14] has proved a duality theorem for certain varieties of weakly associative lattices. His results yield a duality for each $\mathbf{W}_n$ but even in the case $n=3$ his dual space is more complicated than the one introduced here.

As we remarked in the introduction, the duality presented here is essentially an extension of H. A. PRIESTLEY's representation theory for bounded distributive lattices [10, 11]. Firstly, her original theory can be modified slightly so as to cater for distributive lattices with a unit element only [4, Theorem 4.4.7]. Secondly, it is a simple matter to verify that if B is any distributive lattice with a unit element $*$, then its Priestley dual space, in the sense of [4], and its dual space $D(B)$, in the sense of Theorem 2.11, are one and the same thing.

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EXAMPLES OF CONNECTED LEFT-SEPARATED SPACES AND TOPOLOGICAL GROUPS

By

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In this paper we produce two completely regular connected left separated spaces L and G. Our constructions do not depend on any additional set-theoretic assumptions.

The existence of each of these two spaces solves Arhangel'skii's question whether there exists a regular connected left separated space X with $|X|>1$.

Moreover, the spaces L and G have some additional properties. Namely, both are pseudompact and G has the natural structure of a topological group.

The construction of the space L is especially clear. That is why we begin with the construction of the space L and then produce the group G by making in this construction some technical improvements.

Recall that a space Y is called left separated if there exists a well-ordering of $Y=\{y_\alpha:\ \alpha<\tau\}$ such that a left ray $\{y_\beta:\ \beta<\alpha\}$ is closed in Y for each $\alpha<\tau$. This well-ordering is said to be left.

The cardinals are identified with the corresponding ordinals. Each ordinal is considered as the set of all preceding ordinals.

Let us begin with a general lemma which is interesting in itself.

LEMMA. *Let* $X=\prod\limits_{\alpha\in A} K_\alpha$, *where the* K_α *are separable connected compact spaces, and let* R *be a subspace of* X *such that* $\pi_B R=\prod\limits_{\alpha\in B} K_\alpha$ *for every countable subset* $B\subseteqq A$. *Then* R *is a connected pseudocompact space.*

As usual, π_B stands for the natural projection of X onto $\prod\limits_{\alpha\in B} K_\alpha$, where B is an arbitrary subset of A.

PROOF. The condition $\pi_B R=\prod\limits_{\alpha\in B} K_\alpha$, which holds for every countable subset $B\subseteqq A$, implies that R is dense in X. Hence for every continuous real-valued function f defined on R, there exist a countable subset $B\subseteqq A$ and a continuous function g defined on $\pi_B R$ such that $f=g\circ(\pi_B|R)$ (cf. [1], Theorem 1 and the Addition to it). The space $\pi_B R=\prod\limits_{\alpha\in B} K_\alpha$ is compact, thus the continuous function g attains its upper and lower bounds. Since f is the composition of $\pi_B|R$ and g, the above statement is true also for f. So R is pseudocompact.

Now we prove the connectedness of R. Let us assume the contrary. Then there exist non-empty open subsets U, V of X such that $R\subseteqq U\cup V$ and $U\cap R\cap V=\Lambda$. Since R is dense in X, $U\cap V=\Lambda$. One can easily see that the equalities $[U]\cap R=U$ and $[V]\cap R=V$ imply that $[U]\cap[V]\cap R=\Lambda$.

Both sets $[U]$ and $[V]$ being regular closed in a product of separable spaces, are B-marked for some countable subset B of A [2]. (We say that a subset $T\subseteqq X$ is B-marked, for a non-empty $B\subseteqq A$, if $T=\pi_B^{-1}\pi_B T$.) Put $F=[U]\cap[V]$. Then F is B-marked too and $F\cap R=\Lambda$. However the equality $\pi_B R=\prod_{\alpha\in B} K_\alpha$ implies that F is empty.

So $[U]$ and $[V]$ are disjoint nonempty closed subsets of X and $X=[U]\cup[V]$. This contradicts the fact that X (being a product of connected spaces) is connected. The Lemma is proved.

Now we construct the space L. Let I be the unit interval $[0, 1]$ and ε be the set of all ordinals which are less than continuum. Let also Σ be the set of all points $p\in I^\varepsilon$ such that a set $\{\alpha\in\varepsilon: \pi_\alpha p\neq 0\}$ is countable.

THEOREM 1. *There exists a connected, left separated, pseudocompact dense subspace of I^ε contained in Σ. Thus a left separated, pseudocompact completely regular space need not be compact.*

CONSTRUCTION. Let $\{g_\alpha: \alpha\in\varepsilon\}$ be an enumeration of all points of Σ such that every point $p\in\Sigma$ occurs in this sequence ε times. Clearly this is possible, since $|\Sigma|=\varepsilon$. For every $\alpha\in\varepsilon$ we define a point $f_\alpha\in I^\varepsilon$ by letting

$$f_\alpha(\beta)=\begin{cases} g_\alpha(\beta), & \text{if } \beta<\alpha\\ 1, & \text{if } \beta=\alpha\\ 0, & \text{if } \alpha<\beta<\varepsilon.\end{cases}$$

Then it is obvious that $f_\alpha\in\Sigma$ for each $\alpha\in\varepsilon$. Therefore the set $L=\{f_\alpha: \alpha\in\varepsilon\}$ is contained in Σ.

Now we claim that $\pi_B L=I^B$ for every countable subset $B\subset\varepsilon$.

Indeed, let B be a countable subset of ε and p be an arbitrary point of I^B. Then there exists an ordinal $\alpha\in\varepsilon$ such that $\pi_B g_\alpha=p$. Further, the fact that $\omega<\mathrm{cf}(\varepsilon)$ implies the existence of an ordinal $\beta\in\varepsilon$ such that $B\subset\beta$ and $\alpha\leqq\beta$. As the set $T=\{\gamma\in\varepsilon: g_\gamma=g_\alpha\}$ is cofinal in ε, there exists an ordinal $\gamma\in T$ such that $\beta\leqq\gamma$. From the definition of the function f_γ it follows that $\pi_B f_\gamma=\pi_B g_\gamma=p$. Consequently, $\pi_B L=I^B$.

The fact that the space L is left separated follows immediately from the construction. Using the lemma we conclude that the space L is connected and pseudocompact. Thus L possesses all the required properties and the theorem is proved.

REMARK. The space L is not normal. Indeed, otherwise the pseudocompact space L would be countably compact. However, every left separated, countably compact T_2-space is compact [3]. Further, a left separated compact space is scattered ([3], Theorem 1) and every scattered compact T_2-space is zerodimensional [4]. This contradicts the fact that every zerodimensional T_1-space L with $|L|>1$ is not connected.

Hence we raise the following

PROBLEM. Does there exist a normal, connected, left separated space?

To construct the group G we need some preparations.

Let Z be the set of all integers and P be the set of all odd ordinals belonging to ε. Note that any ordinal $\alpha\in P$ is nonlimit. The ordinal 0 is considered to be limit.

Let FA be the free Abelian group generated by the set P. For every element $t\in FA$ we choose a representative word $T(t)$ for t of minimal length. The identity of the group FA has the empty word as its representative. Let $T(t)$ be of the form $\alpha_0^{k_0}\dots\alpha_n^{k_n}$ with different ordinals $\alpha_0, \dots, \alpha_n$ belonging to P and numbers $k_0, \dots, k_n\in Z\setminus\{0\}$. The sum $\sum_{i=0}^{n}|k_i|$ is the length of the word $T(t)$. We put $d(t)=\{\alpha_0, \dots, \alpha_n\}$. The set $d(t)$ is clearly correctly defined, i.e. $d(t)$ does not depend on the choice of the representative word $T(t)$ for t. This fact immediately follows from the minimality of the length of $T(t)$.

We claim that there exists a well-ordering of FA, $FA=\{t_\alpha: \alpha\in\varepsilon\}$ such that

(0) t_0 is the identity of FA;

(1) if α is the representative word for an element $t\in FA$ then $t=t_\alpha$; if α^{-1} is the representative for an element $t\in FA$, then $t=t_{\alpha+1}$.

(2) $d(t_\alpha)\subseteqq\alpha$, for each limit ordinal $\alpha\in\varepsilon$.

(3) for each limit ordinal $\alpha\in\varepsilon$ the set $\{\gamma<\alpha: d(t_\gamma)=d(t_\alpha)\}$ is finite and for every $\beta<\alpha$, either $d(t_\alpha)\setminus d(t_\beta)\neq\Lambda$ or $d(t_\beta)=d(t_\alpha)$.

There are no principal difficulties to enumerate the elements of FA to satisfy the properties (0)—(3). However it is technically involved, therefore we omit the construction of this well-ordering.

Denote by S the circumference with the usual topology and group structure (i.e. $S=\{\varphi: 0\leqq\varphi<2\pi\}$ with addition modulo 2π as the group operation). Let S^ε be the topological group product of ε copies of S. Denote by $\bar{0}$ the element of S^ε every coordinate of which is zero. Let Σ be the set of all $g\in S^\varepsilon$ which coincide with the point $\bar{0}$ on all except countable many coordinates. Then $|\Sigma|=\varepsilon$, so there exists an enumeration $\{g_\alpha: \alpha\in\varepsilon\}$ in which every element $g\in\Sigma$ occurs ε times.

Now we are ready to construct the group G.

Put $f_0=\bar{0}$ and $\mu_0=0$. Now suppose that $1\leqq\alpha<\varepsilon$ and, for each $\beta<\alpha$, an ordinal μ_β with $\beta<\mu_\beta<\beta+\omega$ and a family of maps $\{f_\gamma|\mu_\beta: \gamma\leqq\beta\}$ from μ_β into S have already been defined.

Our construction will be carried out in such a way that the following properties hold:

a) if $\beta\in P$ then $f_\beta(\delta)+f_{\beta+1}(\delta)=0 \pmod{2\pi}$ for each $\delta\in\varepsilon$, and

b) if β is a limit ordinal belonging to ε and $T(t_\beta)=\beta_0^{l_0}\dots\beta_m^{l_m}$ then $f_\beta(\delta)=\sum_{j=0}^{m} l_j\cdot f_{\beta j}(\delta) \pmod{2\pi}$ for each $\delta\in\varepsilon$.

We continue the construction.

I. *α is a nonlimit ordinal.* i) $\alpha\in P$. Then $\alpha=\beta+1$ where $\beta\notin P$. We put $f_\alpha|\mu_\beta=g_\alpha|\mu_\beta$, $f_\alpha(\mu_\beta)=1$ and $f_\gamma(\mu_\beta)=0$ for each $\gamma\leqq\beta$. By putting $\mu_\alpha=\mu_\beta+1$ we have defined the family $\{f_\gamma|\mu_\alpha: \gamma\leqq\alpha\}$ of maps from μ_α into S. It is clear that the point $f_\alpha|\mu_\alpha\in S^{\mu_\alpha}$ does not belong to the closure of the set $\{f_\gamma|\mu_\alpha: \gamma<\alpha\}$ in the space S^{μ_α}.

ii) $\alpha=\beta+1$ *where* $\beta\in P$. Then we put $\mu_\alpha=\mu_\beta+1$ and define the function $f_\alpha|\mu_\alpha$ by the equalities $f_\alpha(\delta)+f_\beta(\delta)=0 \pmod{2\pi}$, $\delta<\mu_\beta$. We also put $f_\alpha(\mu_\beta)=1$ and $f_\beta(\mu_\beta)=2\pi-1$. Finally, put $f_\gamma(\mu_\beta)=0$ for each $\gamma<\beta$.

So we have defined the family $\{f_\gamma|\mu_\alpha: \gamma\leqq\alpha\}$ of maps from μ_α to S and it is clear that the point $f_\alpha|\mu_\alpha\in S^{\mu_\alpha}$ does not belong to the closure of the set $\{f_\gamma|\mu_\alpha: \gamma<\alpha\}$ in the space S^{μ_α}.

II. *α is a limit ordinal* >0. Let $T(t_\alpha)$ be the word $\alpha_0^{k_0}\dots\alpha_n^{k_n}$ with $\alpha_0<\dots<\alpha_n$. Then the properties (0) and (1) of the enumeration of elements belonging to FA implies that $n\geqq 1$. From the property (2) of this enumeration it follows that $\alpha_j<\alpha$ for every $j\leqq n$. For each $i\leqq n$ put $\varepsilon_i=(\varepsilon_{i0},\dots,\varepsilon_{ij},\dots,\varepsilon_{in})$, where $\varepsilon_{ij}=\begin{cases}0 \text{ if } i\neq j\\ 1 \text{ if } i=j.\end{cases}$

As α is a limit ordinal and $\beta<\mu_\beta<\beta+\omega$ for each $\beta<\alpha$, so $\alpha=\sup\{\mu_\beta\colon \beta<\alpha\}$. Consequently, the family of maps $\{f_\beta|\alpha\colon \beta<\alpha\}$ is defined. Put $\mu_\alpha=\alpha+n+1$. Now we have to define the values $f_\beta(\delta)\in S$ for each $\beta<\alpha$ and $\delta\in[\alpha,\mu_\alpha)=\{\gamma\colon \alpha\leqq\gamma<\mu_\alpha\}$.

We also will define the function $f_\alpha|\mu_\alpha$.

Put $\gamma_i=\alpha+i$ for each $i\leqq n$. For each $i,j\leqq n$ we put $f_{\alpha_j}(\gamma_i)=\varepsilon_{ij}$. For every non-limit ordinal $\beta\in\alpha\setminus d(t_\alpha)$ and every $i\leqq n$ put $f_\beta(\gamma_i)=0$. Then we put $f_\alpha(\delta)=\sum_{j=0}^{n}k_j\cdot f_{\alpha_j}(\delta) \pmod{2\pi}$ for each $\delta<\mu_\alpha$. For every limit ordinal $\beta\in(0,\alpha)$ the function f_β on the segment $[\alpha,\mu_\alpha)$ is defined analogously to the function f_α. Namely, let $T(t_\beta)=\beta_0^{l_0}\dots\beta_m^{l_m}$. Then we put $f_\beta(\delta)=\sum_{j=0}^{m}l_j\cdot f_{\beta_j}(\delta) \pmod{2\pi}$ for every $\delta\in[\alpha,\mu_\alpha)$.

So the family of functions $\{f_\gamma|\mu_\alpha\colon \gamma\leqq\alpha\}$ is defined completely. Now we claim that the point $f_\alpha|\mu_\alpha\in S^{\mu_\alpha}$ does not belong to the closure of the set $\{f_\gamma|\mu_\alpha\colon \gamma<\alpha\}$ in the space S^{μ_α}.

Indeed, the set $A=\{\beta<\alpha\colon d(t_\beta)=d(t_\alpha)\}$ is finite because of the property (3) of the enumeration $\{g_\alpha\colon \alpha\in\varepsilon\}$. Let $\beta\in A$ and $T(t_\beta)=\beta_0^{l_0}\dots\beta_n^{l_n}$ with $\beta_0<\dots<\beta_n$. Since $t_\alpha\neq t_\beta$, there exists $i\leqq n$ such that $k_i\neq l_i$. It is easily seen then that $f_\beta(\gamma_i)=l_i\neq k_i=f_\alpha(\gamma_i)$. We must only note that $k\neq 0 \pmod{2\pi}$ for every $k\in Z\setminus\{0\}$ hence $f_\beta(\gamma_i)\neq f_\alpha(\gamma_i) \pmod{2\pi}$. So $f_\beta|\mu_\alpha\neq f_\alpha|\mu_\alpha$ for any $\beta\in A$. Using the finiteness of the set A one can evidently find an open neighbourhood V of the point $f_\alpha|\mu_\alpha$ in the space S^{μ_α} such that $V\cap\{f_\gamma|\mu_\alpha\colon \gamma\in A\}=\Lambda$.

Now let us take $\beta\in\alpha\setminus A$. Then $d(t_\alpha)\setminus d(t_\beta)\neq\Lambda$. Let $T(t_\beta)=\beta_0^{l_0}\dots\beta_m^{l_m}$. Then there exists $i\leqq n$ such that $\alpha_i\neq\beta_j$ for each $j\leqq m$. It is obvious then that $f_\beta(\gamma_i)=0\neq k_i=f_\alpha(\gamma_i)$. The last inequality implies that the open neighbourhood $W=\{f\in S^{\mu_\alpha}\colon f(\gamma_i)\neq 0 \text{ for each } i\leqq n\}$ of the point $f_\alpha|\mu_\alpha\in S^{\mu_\alpha}$ does not intersect the set $\{f_\beta|\mu_\alpha\colon \beta\in\alpha\setminus A\}$.

So the open neighbourhood $V\cap W$ of the point $f_\alpha|\mu_\alpha$ in the space S^{μ_α} does not intersect the set $\{f_\gamma|\mu_\alpha\colon \gamma<\alpha\}$. This completes our recursive construction. It is easily seen that conditions (a) and (b) of the construction are satisfied.

Put $G=\{f_\alpha\colon \alpha\in\varepsilon\}\subseteqq S^\varepsilon$.

Consider the topology on G induced from S^ε. The well-ordering $\{f_\alpha\colon \alpha\in\varepsilon\}$ on G is left; this follows from our construction.

It is obvious that $\bar{0}\in G$. Moreover, for every $\alpha\in\varepsilon$ there exists $\beta\in\varepsilon$ such that $f_\alpha+f_\beta=\bar{0}$ (addition is taken in the group S^ε). For $\alpha\in P$ this follows from the property (1) of the enumeration of elements belonging to FA and condition (a) of our recursive construction. One can analogously verify this statement for limit $\alpha\in\varepsilon$.

Finally, for any $\alpha,\beta\in\varepsilon$ the element $f_\alpha+f_\beta\in S^\varepsilon$ belongs to G. Indeed, for every $\alpha,\beta\in\varepsilon$ there exists an element $t_\gamma\in FA$ such that $t_\alpha\cdot t_\beta=t_\gamma$. Condition (b) of the construction implies the equality $f_\gamma=f_\alpha+f_\beta$.

So the space G has the natural structure of a topological group (as a subgroup of S^ε). The verification of the fact that $\pi_B G=S^B$ for each countable $B\subset\varepsilon$ can be carried out analogously as for the space L. Consequently, the group G is connected and pseudocompact. Moreover the Stone—Čech compactification βG of the group G is topologically isomorphic to S^ε.

Note that the bijection $t_\alpha\mapsto f_\alpha$ is an algebraic isomorphism of the group FA onto G.

Problem. Is it possible to construct a topological group having the properties of the group G and lying in a Σ-product of circumferences?

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CORRECTION OF SOME MISPRINTS IN OUR PAPER*

Page	Line	Read	Instead of
74	7	$1/\ln\ln m$; further	$1/\ln\ln m$. Further
75	11	$2\sqrt{\ln n}\,\delta_n$	$\sqrt{\ln n}\,\delta_n$
75	21	$\geqq \lvert\omega(\bar{z}_k)\rvert$	$\geqq \omega(\bar{z}_k)$
76	13	Dropping J_t	Dropping J_l
79	3	$(n \geqq n_1(q))$	$(n \geqq n_0(q))$
80	7	$\sum_{t=1}^{s} f(x_{k_t}) l_{k_t}(x)$	$\sum_{t=1}^{n} f(x_{k_s}) l_{k_s}(x)$
81	(4.29)	$\sum_{t=1}^{\infty} \dots$	$\sum_{t=k}^{\infty} \dots$
83	24	that is	and that
86	(4.57)	$W = \bigcap_{k=1}^{\infty} \bigcup_{t=k}^{\infty} \dots$	$W = \bigcup_{k=1}^{\infty} \bigcup_{t=k}^{\infty} \dots$
87	5	by $W_{tl} = R_{tl}^{[0]}$, $W = G^{[0]}$	by $W = G^{[0]}$
87	6	$G_\varrho \cup W$	$G_\varrho \cap W$
88	3	ε and $M (M \geqq 1$, integer)	ε and M
88	10	$\mu\left(\bigcap_{t=0}^{\infty} H_t\right) \geqq \varepsilon$, which	$\mu\left(\bigcup_{t=0}^{\infty} H_t\right) \geqq \varepsilon$ wich
88	15		

Read:

$$``\lvert L_{u_1(x)}(f_1, x)\rvert \geqq A_1 > 1^3\lambda_{N_0}^2 \quad \text{whenever} \quad x \in S_1.$$

Here $m_1 \leqq u_1(x) \leqq n_1$. Now we take the polynomial $\varphi_1(f_1, x)$ of degree $\leqq N_1$ $\|\varphi_1\| \leqq 32$, for which

$$\lvert L_{u_1(x)}(\varphi_1, x)\rvert \geqq A_1 > 1^3\lambda_{N_0}^2 \quad \text{whenever} \quad x \in S_1$$

(see 4.4.4)."

instead of:

$$``\lvert L_{u_1(x)}(\varphi_1, x)\rvert \geqq A_1 > 1^3\lambda_{N_0}^2 \quad \text{whenever} \quad x \in S_1$$

(see 4.4.4)."

88	16	$\varrho_k = 2^{-k}$, $A_k > k^8\lambda_{N_{k-1}}^2$	$\delta_k = 2^{-k}$, $A_k > k_\varepsilon\lambda_{N_{k-1}}^2$
88	18	$\geqq 2 - 2\varrho_k$	$\geqq 2 - 2\delta_k$
88	23	$S = \bigcap_{k=1}^{\infty} \bigcup_{i=k}^{\infty} S_i$	$S = \bigcap_{k=1}^{\infty} \bigcap_{i=k}^{\infty} S_i$

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* P. Erdős and P. Vértesi, On the almost everywhere divergence of Lagrange interpolatory polynomials for arbitrary system of nodes, *Acta Math. Acad. Sci. Hungar.*, 36 (1980), 71—89.

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